ON Sp_4 MODULARITY OF PICARD–FUCHS DIFFERENTIAL EQUATIONS FOR CALABI–YAU THREEFOLDS

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Abstract. Motivated by the relationship of classical modular functions and Picard–Fuchs linear differential equations of order 2 and 3, we present an analogous concept for equations of order 4 and 5.

0. Introduction

Let M_z be a family of Calabi–Yau threefolds parameterized by a complex variable $z \in \mathbb{P}^1(\mathbb{C})$. Then periods of the unique holomorphic differential 3-form on M_z satisfy a linear differential equation, called the *Picard–Fuchs differential equation* of M_z. When the Hodge number $h^{2,1}$ is equal to 1, the Picard–Fuchs differential equation has order 4. One of the most well-known examples is perhaps the family of quintic threefolds

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - z^{-1/5} x_1 x_2 x_3 x_4 x_5 = 0$$

in $\mathbb{P}^4$, whose Picard–Fuchs differential equation is

$$\theta^4 y - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)y = 0, \qquad \theta = z\frac{\mathrm{d}}{\mathrm{d}z} \tag{1}$$

(see [5]). This is one of the fourteen families of Calabi–Yau threefolds whose Picard–Fuchs differential equations are hypergeometric (we refer the reader to the classical book [16] for the definition of hypergeometric functions and hypergeometric differential equations).

Very recently, we [6] studied the monodromy aspect of Picard–Fuchs differential equations originated from Calabi–Yau threefolds. One of the main results in [6] is that, with respect to certain bases, the monodromy groups of the fourteen hypergeometric Picard–Fuchs differential equations are contained in certain congruence subgroups of $\mathrm{Sp}_4(\mathbb{Z})$ (see [6, Theorem 2]). For instance, in the case of (1), the monodromy matrices around the singular points $z = 0$ and $z = 1/3125$ are

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

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respectively. The group generated by these two matrices is contained in the congruence subgroup

$$\left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \pmod 5 \right\}.$$

The numerical computation in [6] suggests that a similar phenomenon also occurs in other non-hypergeometric cases. This naturally leads us to the question whether the Picard–Fuchs differential equations for Calabi–Yau threefolds are related to Siegel modular forms in some way.

To be specific, recall the classical result that the solution ${}_2F_1(1/2,1/2;1;z)$ of the Picard–Fuchs differential equation

$$\theta^2 y - \frac{z}{4}(2\theta+1)^2 y = 0$$

for the family

$$E_z : y^2 = x(x-1)(x-z)$$

of elliptic curves (i.e., of Calabi–Yau onefolds) satisfies

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{\theta_2^4}{\theta_3^4}\right) = \theta_3^2,$$

where $\theta_2(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau(n+1/2)^2}$ and $\theta_3(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2}$ are modular forms of weight $1/2$. In other words, under a suitable setting, z becomes a modular function and the holomorphic solution of the differential equation at $z=0$ becomes a modular form of weight 1 on the congruence subgroup $\Gamma(2)$ of $\mathrm{SL}_2(\mathbb{Z})$ (see Section 1 for a detailed account of this interpretation). Likewise, the solution ${}_3F_2(1/4,1/2,3/4;1,1;256z)$ of the Picard–Fuchs differential equation

$$\theta^3 y - 4z(4\theta+1)(4\theta+2)(4\theta+3)y = 0 \tag{3}$$

for the family

$$K_z : x_1^4 + x_2^4 + x_3^4 + x_4^4 - z^{-1/4}x_1x_2x_3x_4 = 0$$

of $K3$ surfaces (i.e., of Calabi–Yau twofolds) can be interpreted as a modular form of weight 2 on $\Gamma_0^+(2)$ under a suitable setting. Therefore, one might be tempted to conjecture that the holomorphic solution of (1) at $z=0$ can be interpreted as a Siegel modular form. The main purpose of the present article is to address this modularity question.

It turns out that there are several ways to give Sp_4-modular interpretation, although none of which gives a direct link to Siegel modular forms. The first of them is given in Section 2, in which we will show that each fourth order Picard–Fuchs differential equation for a family of Calabi–Yau threefolds can be associated with a fifth order linear differential equation, whose holomorphic solution $w(z)$ at $z=0$, under a suitable formulation, transforms like a Siegel modular form of weight 1 under the action of the monodromy group. This result can be regarded as a generalization of the classical Schwarz theory of second order linear differential equations and automorphic functions.

The formulation of the second modular interpretation is originally due to A. Klemm et al [1], [10]. Using geometric insights, they showed that the parameter space of

a Picard–Fuchs differential equation of a family of Calabi–Yau threefolds can be embedded into the Siegel upper half-space

$$\mathbb{H}_2 = \{\mathrm{Z} \in M_2(\mathbb{C}) : {}^t\mathrm{Z} = \mathrm{Z},\ \mathrm{Im}\,\mathrm{Z} > 0\},$$

albeit non-holomorphically. Furthermore, they showed that the image of the embedding transforms under the action of monodromy in the same way as $\mathbb{H}_2$ does under the action of $\mathrm{Sp}_4(\mathbb{R})$. In Section 3, we will extend this geometric formulation to general fourth order differential equations with symplectic monodromy. We will show that under some conditions on the solution space of the differential equations, we can similarly embed non-holomorphically the parameter space of the differential equation into $\mathbb{H}_2$. (If the differential equation is coming from geometry, the conditions are fulfilled in view of the argument in [1], [10].) Moreover, we will show that if we modify the function $w(z)$ in the previous paragraph by a certain non-holomorphic factor, then the resulting function transforms like a Siegel modular form of weight 1 under this formulation.

We stress that, in either interpretation, the functions may not be related to a true Siegel modular form at all. There are several reasons for this. One is that the monodromy groups may not be of finite index in $\mathrm{Sp}_4(\mathbb{Z})$. (A proof of infiniteness for a particular example is provided by V. Paşol in the Appendix. As pointed out to us by D. van Straten, the infiniteness of index for the group generated by the matrices in (2) might be deduced from the Margulis–Tits theorem [15, Chap. I, § 6.6] modulo some unproved observation in [5].) Also, for the first formulation, the domain on which the function w is defined, is not always in the Siegel upper half-space. In fact, the discussion in Section 3 shows that for Klemm–et al's embedding in the second formulation to be inside $\mathbb{H}_2$, the domain in the first formulation has to be disjoint from $\mathbb{H}_2$. It is a complete mystery how our functions are related to Siegel modular forms (if they are).

In Section 4 we consider the converse problems. Recall that in the classical Schwarz theory, a second order linear differential equation with nice monodromy gives rise to a modular function and a modular form of weight 1 and, conversely, for each pair of a modular form of weight 1 and a non-constant modular function, there associates a second order linear differential equation. In Section 4 we develop an analogous theory for fourth order Picard–Fuchs differential equations with monodromy inside the symplectic group.

The rest of this article is organized as follows. In Section 1 we review the classical theory of second order linear differential equations and automorphic functions. This will serve the purpose as a guideline for our development of a corresponding theory for fourth order Calabi–Yau differential equations. In Sections 2 and 3 we describe the modular interpretations of Calabi–Yau equations mentioned above. In Section 4 we state the converse results and give some examples in details. The proof of the converse results will be given in Section 5. Finally, in Section 6 we present some arithmetic observations on a concrete example of a fifth order Picard–Fuchs linear differential equation.

1. Modular Picard–Fuchs equations

The classical theory concerns with second order linear differential equations having rational coefficients and regular singularities at $z = 0, \infty$ and some other points. Let us fix such an equation and assume that its projective monodromy group Γ is

a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ commensurable with the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Then we may choose two linearly independent solutions $u_0 = u_0(z)$ and $u_1 = u_1(z)$ such that

$$\gamma\colon \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \mapsto \chi(\gamma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \qquad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{4}$$

where $\chi(\gamma)$ is a root of unity depending on γ, and the image (of a suitably cut $\mathbb{C}$-plane) under the multivalued map $\tau(z) = u_1(z)/u_0(z)$ fills the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$.

Remark 1. If the original differential equation has the both exponents to be 0 at the origin, the usual choice for a pair of solutions is $u_0 = u_0(z) \in 1 + z\mathbb{C}[[z]]$ and $u_1 = u_1(z) = \frac{\sqrt{D}}{2\pi i}(u_0(z)\log z + v(z))$ for some $D \in \mathbb{Q}$ (normally, $D = 1$) and $v(z) \in z\mathbb{C}[[z]]$. In this case, the images of the singular point $z = 0$ under the map $\tau(z)$ form a set of cusps.

The following is an immediate consequence of our settings and the Schwarz theory.

Theorem A. *The inverse $z = z(\tau)$ of the map $\tau(z) = u_1(z)/u_0(z)$ is a (meromorphic) Γ-modular function (of weight 0). The function u_0 viewed as a function of the variable τ is a Γ-'modular' form of weight 1.*

Remark 2. We quote the word 'modular' since u_0 is not necessarily holomorphic in $\mathbb{H}\cup\{\text{cusps}\}$: it may have branching (of finite degree) at elliptic points. Nevertheless, a suitable choice of a positive integer N gives us a holomorphic Γ-modular form u_0^N of weight N.

Proof. Indeed, the invariance of $z(\tau)$ under the action of Γ follows from the definition, while from (4) we see that

$$\gamma\colon u_0 \mapsto \chi(\gamma)(cu_1 + du_0) = \chi(\gamma)(c\tau + d)\cdot u_0 \qquad \text{for all} \quad \gamma \in \Gamma,$$

where $\chi(\gamma)$ is a root of unity depending on γ. $\square$

Therefore, any meromorphic Γ-modular form of weight 0 is an algebraic function of $z(\tau)$, while any meromorphic Γ-modular form of weight k may be represented as $g(z)u_0^k$, where g is an algebraic function.

Corollary. *The wronskian $W(u_0, u_1) = u_0u_1' - u_0'u_1$ is an algebraic function of z.*

Modular proof. It follows from (4) that

$$\begin{aligned} \gamma\colon W(u_0, u_1) \mapsto W(cu_1 + du_0, au_1 + bu_0) &= (ad - bc)W(u_0, u_1) \\ &= W(u_0, u_1) \qquad \text{for all} \quad \gamma \in \Gamma, \end{aligned}$$

hence $W(u_0, u_1)$ is a Γ-modular form of weight 0. This, in particular, implies the required statement. $\square$

Analytic proof. Let $u'' + A(z)u' + B(z)u = 0$ denote the original differential equation satisfied by u_0 and u_1. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}W(u_0, u_1) = (u_0u_1' - u_0'u_1)' = u_0u_1'' - u_0''u_1 = -A(z)W(u_0, u_1).$$

Since the differential equation is of Picard–Fuchs type, the rational function $A(z)$ may be decomposed into the following sum of partial fractions:

$$A(z) = \sum_{k=1}^{K} \frac{A_k}{z - z_k}, \qquad A_k \in \mathbb{C},$$

where $z_1, \dots, z_K$ are the finite singularities of the original equation. It remains to integrate the equation $\mathrm{d}W/\mathrm{d}z = -A(z)W$ to make certain of the algebraicity of W as a function of z. □

For further use, write

$$W(u_0, u_1) = \frac{1}{Cg_0(z)}, \tag{5}$$

where g_0 is an algebraic function (that can be explicitly evaluated for the given second order linear differential equation) and $C \neq 0$ is a certain normalization constant (for instance, in the case considered in Remark 1 the constant C is usually taken to be $2\pi i$).

This can be also written as

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = \frac{W(u_0, u_1)}{u_0^2} = \frac{1}{Cg_0 u_0^2}, \tag{6}$$

thus showing that $\mathrm{d}z/\mathrm{d}\tau$ is a Γ-'modular' form of weight 2.

It is interesting that, if we similarly start with a third order Picard–Fuchs differential equation, it always comes as the (symmetric) square of a second order equation of the form described above. The projective monodromy group of the third order equation is a discrete subgroup of $O_3(\mathbb{R})$ commensurable with $O_3(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z})$. This means that we are in about the same case as above, and an (analytic) solution of the differential equation is a 'modular' form of weight 2.

The situation changes drastically if the differential equation for a function $u_0(z)$ has order $l \geq 4$: it may be the $(l-1)$-st power of a second order linear differential equation (when we are faced again to the case discussed above) but there are plenty of other possibilities in the general case. Note that in the modular case, i.e., in the case of the reduction to a second order equation, we have some kind of the inverse statement:

Theorem B (folklore; for three different proofs see [17, Subsec. 5.4])**.** *Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$, $z = z(\tau)$ a non-constant meromorphic Γ-modular form of weight 0 and $U_0 = U_0(\tau)$ a meromorphic Γ-modular form of weight $l-1$. Then $U_0, \tau U_0, \dots, \tau^{l-1}U_0$ viewed as functions of z are linearly independent solutions of an l-th order linear differential equation whose coefficients are algebraic functions of z.*

In what follows we give analogs of Theorems A and B for linear differential equations of order 4 and 5 in the case when the projective monodromy group Γ is commensurable with a discrete subgroup of $\mathrm{Sp}_4(\mathbb{Z}) \simeq O_5(\mathbb{Z})$.

2. Differential equations with monodromy Sp_4

Consider now a fourth order Picard–Fuchs differential equation with projective monodromy group $\Gamma \subset \mathrm{Sp}_4(\mathbb{R})$ commensurable with a discrete subgroup of $\mathrm{Sp}_4(\mathbb{Z})$

(of not necessarily finite index). Assume that there is a point of maximally unipotent monodromy. Gather its fundamental matrix solution

$$\begin{pmatrix} u_3 & u_3' & u_3'' & u_3''' \\ u_2 & u_2' & u_2'' & u_2''' \\ u_1 & u_1' & u_1'' & u_1''' \\ u_0 & u_0' & u_0'' & u_0''' \end{pmatrix}, \tag{7}$$

where the basis u_0, u_1, u_2, u_3 is chosen in such a way that

$$W(u_0, u_2) + W(u_1, u_3) = 0 \tag{8}$$

and the monodromy matrices are in Γ. Introduce the functions

$$w_{jl} = CW(u_j, u_l) = C(u_j u_l' - u_j' u_l), \quad w_{jl} = -w_{lj}, \qquad 0 \le j, l \le 3,$$

where $C \ne 0$ is a certain normalization constant. Thanks to (8) we have a linear relation

$$w_{02} + w_{13} = 0; \tag{9}$$

there is also a quadratic relation

$$w_{01}w_{23} + w_{02}w_{13} + w_{03}w_{12} = 0,$$

which is tautological in terms of the u_js. The five linearly independent functions

$$w_{01}, \quad w_{02} = -w_{13}, \quad w_{03}, \quad w_{12}, \quad w_{23}$$

form a solution to a fifth order linear differential equation (the so-called *antisymmetric square*) with the monodromy conjugate to a subgroup commensurable to a discrete subgroup of $O_5(\mathbb{Z}) \simeq \mathrm{Sp}_4(\mathbb{Z})$.

Remark 3. Assuming additionally that the local exponents of the equation at the origin are all zero we may always choose the above basis such that $u_0 \in 1 + z\mathbb{C}[[z]]$ is the analytic solution at the origin, while $u_1 = \frac{1}{2\pi i}(u_0 \log z + v)$ for some $v \in z\mathbb{C}[[z]]$. The inverse $z(t)$ of the mapping $t(z) = u_1(z)/u_0(z)$ is known in the theory of Calabi–Yau threefolds as the *mirror map*. Note that in this special case we take $C = 2\pi i$ and the solution w_{01} of the above fifth order equation becomes analytic at the origin after multiplication by z by our choice of u_0 and u_1.

Remark 4. We remark that relation (8) is a feature somehow unique to differential equations with a point of maximally unipotent monodromy. According to the general differential Galois theory (see [13]), if the differential Galois group (the Zariski closure of the monodromy group) of a fourth order linear differential equation with regular singularities is $\mathrm{Sp}_4(\mathbb{C})$, then there exists a basis u_0, u_1, u_2, u_3 for the solution space such that $W(u_0, u_2) + W(u_1, u_3)$ is invariant under the monodromy group. However, this function may not be identically zero in general. For instance, the Picard–Fuchs differential equation attached to the operator

$$\theta^2\Big(\theta - \frac{1}{3}\Big)\Big(\theta + \frac{1}{3}\Big) - z\Big(\theta + \frac{1}{2}\Big)^2\Big(\theta + \frac{5}{6}\Big)\Big(\theta + \frac{7}{6}\Big)$$

of the family $y^2 = x(x-1)(x^3 - z)$ of hyperelliptic curves has $\mathrm{Sp}_4(\mathbb{C})$ as its differential Galois group, according to the criteria of Beukers and Heckman [4]. However, the exterior square of this differential equation has order 6. In other words, $W(u_0, u_2) + W(u_1, u_3)$ is not the zero function. Instead, it is equal to $1/(z(1-z))$ (with a suitably chosen basis).

On the other hand, if a fourth order differential equation with $\mathrm{Sp}_4(\mathbb{C})$ as its differential Galois group has a point of maximally unipotent monodromy, then the exterior square also has maximally unipotent monodromy at the same singularity. Since the highest power of $\log z$ in $W(u_i, u_j)$ is at most 4, the exterior square has order at most 5. This forces relation (8) to exist.

Monodromy matrices $\gamma \in \Gamma$ act by left matrix multiplication:

$$\gamma\colon \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \\ u_1 & u_1' \\ u_0 & u_0' \end{pmatrix} \mapsto \gamma \cdot \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \\ u_1 & u_1' \\ u_0 & u_0' \end{pmatrix}. \tag{10}$$

The matrix

$$\begin{aligned} \mathrm{T} = \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \end{pmatrix} \begin{pmatrix} u_1 & u_1' \\ u_0 & u_0' \end{pmatrix}^{-1} &= \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \end{pmatrix} \begin{pmatrix} -u_0' & u_1' \\ u_0 & -u_1 \end{pmatrix} \frac{1}{u_0u_1' - u_0'u_1} \\ &= \begin{pmatrix} w_{03}/w_{01} & w_{31}/w_{01} \\ w_{02}/w_{01} & w_{21}/w_{01} \end{pmatrix} \end{aligned} \tag{11}$$

is symmetric since $w_{31} = -w_{13} = w_{02}$. Denote

$$\tau_1(z) = \frac{w_{03}}{w_{01}}, \quad \tau_2(z) = \frac{w_{02}}{w_{01}} = \frac{-w_{13}}{w_{01}}, \quad \tau_3(z) = \frac{-w_{12}}{w_{01}}, \tag{12}$$

hence

$$\mathrm{T} = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}, \qquad \det \mathrm{T} = \frac{w_{23}}{w_{01}}. \tag{13}$$

Then we have the standard Sp_4-action on this 2×2 symmetric matrix:

$$\begin{aligned} \gamma\colon \mathrm{T} &= \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \end{pmatrix} \begin{pmatrix} u_1 & u_1' \\ u_0 & u_0' \end{pmatrix}^{-1} \\ &\mapsto \left(A \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \end{pmatrix} + B \begin{pmatrix} u_1 & u_1' \\ u_0 & u_0' \end{pmatrix} \right) \left(C \begin{pmatrix} u_3 & u_3' \\ u_2 & u_2' \end{pmatrix} + D \begin{pmatrix} u_1 & u_1' \\ u_0 & u_0' \end{pmatrix} \right)^{-1} \\ &= (A\mathrm{T} + B)(C\mathrm{T} + D)^{-1} = \gamma\mathrm{T} \qquad \text{for} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \end{aligned}$$

Note that the entries of $\mathrm{T} = \mathrm{T}(z)$ in (13) are algebraically independent over $\mathbb{C}(z)$. This follows from the structure of algebraic relations over $\mathbb{C}(z)$ in the fundamental matrix solution (7); the relations are induced by the differential Galois group of the starting fourth order equation and the latter group is isomorphic to the Zariski closure in $\mathrm{GL}_4(\mathbb{C})$ of the monodromy group Γ (see, e.g., [13, Section 5.1]), hence is in $\mathrm{Sp}_4(\mathbb{C})$.

The multivalued function $\tau = \tau_1(z)$ takes values in a certain domain $H \subset \mathbb{C}$. Viewing T as a matrix-valued function of τ, we will say that a function $f(\mathrm{T}(\tau))\colon H \to \mathbb{C}$ is a Γ-*modular form of weight* k if

$$f(\gamma\mathrm{T}) = \det(C\mathrm{T} + D)^k \cdot f(\mathrm{T}) \qquad \text{for all} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

Theorem 1. *The inverse $z = z(\mathrm{T}(\tau))$ of the map $\tau = \tau_1(z)$ in* (12), (13) *is a Γ-modular form of weight* 0. *Furthermore, the function w_{01} viewed as a function of $\mathrm{T} = \mathrm{T}(\tau)$ is a Γ-modular form of weight* 1.

Proof. The invariance of z under the action of Γ is a consequence of its definition.

Since

$$\begin{aligned}\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}:\begin{pmatrix}u_1&u_1'\\u_0&u_0'\end{pmatrix}&\mapsto C\begin{pmatrix}u_3&u_3'\\u_2&u_2'\end{pmatrix}+D\begin{pmatrix}u_1&u_1'\\u_0&u_0'\end{pmatrix}\\&=(C\mathrm{T}+D)\cdot\begin{pmatrix}u_1&u_1'\\u_0&u_0'\end{pmatrix}\end{aligned}\tag{14}$$

(see (10)), taking the determinants of the both sides (and normalizing by $-C$) we obtain

$$\gamma\colon w_{01}\mapsto\det(C\mathrm{T}+D)\cdot w_{01}.$$

This shows that w_{01} is a Γ-modular form of weight 1. □

We now proceed with further computations. Let us compute z-derivatives of τ_j in (12). We have

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}z}=\frac{w_{03}'w_{01}-w_{03}w_{01}'}{w_{01}^2}$$

and for the latter numerator,

$$\begin{aligned}&\frac{w_{03}'w_{01}-w_{03}w_{01}'}{(2\pi i)^2}\\&\quad=(u_0u_3''-u_0''u_3)(u_0u_1'-u_0'u_1)-(u_0u_3'-u_0'u_3)(u_0u_1''-u_0''u_1)\\&\quad=u_0W(u_0,u_1,u_3),\end{aligned}$$

hence

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}z}=\frac{u_0W(u_0,u_1,u_3)}{w_{01}^2}.$$

In the similar way,

$$\begin{aligned}\frac{\mathrm{d}\tau_2}{\mathrm{d}z}&=\frac{w_{01}'w_{13}-w_{01}w_{13}'}{w_{01}^2}=\frac{-u_1W(u_0,u_1,u_3)}{w_{01}^2}\\&=\frac{w_{02}'w_{01}-w_{02}w_{01}'}{w_{01}^2}=\frac{u_0W(u_0,u_1,u_2)}{w_{01}^2},\\\frac{\mathrm{d}\tau_3}{\mathrm{d}z}&=\frac{w_{01}'w_{12}-w_{01}w_{12}'}{w_{01}^2}=\frac{-u_1W(u_0,u_1,u_2)}{w_{01}^2}.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}z}:\frac{\mathrm{d}\tau_2}{\mathrm{d}z}:\frac{\mathrm{d}\tau_3}{\mathrm{d}z}=1:\left(-\frac{u_1}{u_0}\right):\left(-\frac{u_1}{u_0}\right)^2=1:(-t):t^2,\tag{15}$$

where $t=t(z)=u_1(z)/u_0(z)$ (the mirror map in the case of Remark 3).

Corollary. *The function*

$$U=\det\begin{pmatrix}u_0&u_2\\u_0'''&u_2'''\end{pmatrix}+\det\begin{pmatrix}u_1&u_3\\u_1'''&u_3'''\end{pmatrix}\tag{16}$$

is an algebraic function of z, say,

$$U=-\frac{1}{Cg_0(z)}\tag{17}$$

for an algebraic function g_0. Moreover, we have the identity

$$W(u_0,u_1,u_3)=\frac{u_0}{2\pi ig_0(z)}.\tag{18}$$

Proof. Differentiating twice equality (8) we obtain

$$\det\begin{pmatrix} u_0 & u_2 \\ u_0'' & u_2'' \end{pmatrix} + \det\begin{pmatrix} u_1 & u_3 \\ u_1'' & u_3'' \end{pmatrix} = 0 \tag{19}$$

and

$$\det\begin{pmatrix} u_0 & u_2 \\ u_0''' & u_2''' \end{pmatrix} + \det\begin{pmatrix} u_1 & u_3 \\ u_1''' & u_3''' \end{pmatrix} + \det\begin{pmatrix} u_0' & u_2' \\ u_0'' & u_2'' \end{pmatrix} + \det\begin{pmatrix} u_1' & u_3' \\ u_1'' & u_3'' \end{pmatrix} = 0. \tag{20}$$

In particular, the last equality gives us the expression

$$U = -\det\begin{pmatrix} u_0' & u_2' \\ u_0'' & u_2'' \end{pmatrix} - \det\begin{pmatrix} u_1' & u_3' \\ u_1'' & u_3'' \end{pmatrix}. \tag{21}$$

Summing (16), (21) and differentiating we arrive at the equality

$$2U' = \det\begin{pmatrix} u_0 & u_2 \\ u_0'''' & u_2'''' \end{pmatrix} + \det\begin{pmatrix} u_1 & u_3 \\ u_1'''' & u_3'''' \end{pmatrix}. \tag{22}$$

Using the original fourth order linear differential equation $u^{(4)} + A(z)u^{(3)} + \dots = 0$, equalities (8), (19) and definition (16) we finally find from (22) that

$$2U' = -A(z)U. \tag{23}$$

It remains to repeat the argument given in the analytic proof of Corollary of Theorem A to get the algebraicity of U as a function of z.

Expanding the determinant

$$W(u_0, u_1, u_3) = \det\begin{pmatrix} u_0 & u_1 & u_3 \\ u_0' & u_1' & u_3' \\ u_0'' & u_1'' & u_3'' \end{pmatrix}$$

along the first column and using then equalities (8), (19) and (21) we obtain

$$\begin{aligned} W(u_0, u_1, u_3) &= u_0 \det\begin{pmatrix} u_1' & u_3' \\ u_1'' & u_3'' \end{pmatrix} - u_0' \det\begin{pmatrix} u_1 & u_3 \\ u_1'' & u_3'' \end{pmatrix} + u_0'' \det\begin{pmatrix} u_1 & u_3 \\ u_1' & u_3' \end{pmatrix} \\ &= -u_0 \left(U + \det\begin{pmatrix} u_0' & u_2' \\ u_0'' & u_2'' \end{pmatrix} \right) \\ &\qquad + u_0' \det\begin{pmatrix} u_0 & u_2 \\ u_0'' & u_2'' \end{pmatrix} + u_0'' \det\begin{pmatrix} u_0 & u_2 \\ u_0' & u_2' \end{pmatrix} \\ &= -u_0 U - W(u_0, u_0, u_2) = -u_0 U = \frac{u_0}{Cg_0(z)}. \end{aligned}$$

This proves (18). □

Recalling that $t(z) = u_1(z)/u_0(z)$ we finally get the formulas

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{w_{01}}{Cu_0^2}, \qquad \frac{\mathrm{d}\tau_1}{\mathrm{d}z} = \frac{u_0^2}{Cg_0 w_{01}^2}, \tag{24}$$

hence

$$\frac{\mathrm{d}t}{\mathrm{d}z} \cdot \frac{\mathrm{d}\tau_1}{\mathrm{d}z} = \frac{1}{C^2 g_0 w_{01}}.$$

The resulted formulas may be viewed as analogs of formula (6). (Freely speaking, the product $(\partial z/\partial t) \cdot (\partial z/\partial \tau_1)$ is a Γ-'modular' form of weight 1.)

Remark 5. Some time ago, the first author communicated that the antisymmetric square of the fifth order differential equation equals the symmetric square of the corresponding fourth order differential equation. This became later the subject of Section 2 in [2] (see also Theorem 5 below). The theorem easily allows one to deduce that $w_{03}'w_{01} - w_{03}w_{01}' = C^2 u_0 W(u_0, u_1, u_3)$ differs from u_0^2 by an algebraic function of z, which is equivalent to the above corollary.

3. A non-holomorphic Sp_4-modularity

In this section, we discuss another Sp_4-modular interpretation for fourth order Picard–Fuchs differential equations associated with Calabi–Yau threefolds. The formulation is given in [1, Section 5] and [10, Section 6]. Throughout the section we will retain all the notation used in the previous section.

Let u_0, u_1, u_2, u_3 be a basis of the solution space of the Picard–Fuchs differential equation for a family of Calabi–Yau threefolds such that the monodromy group with respect to the basis is $\Gamma \subset \mathrm{Sp}_4(\mathbb{R})$ (the recipe we follow in the previous section). Consider, as in [1, Section 5], the function

$$F = \frac{u_0u_2 + u_1u_3}{2u_0^2}$$

and let

$$\tau_{ij} = \frac{\partial^2 F}{\partial u_i \partial u_j}$$

for $i, j = 0, 1$. Set

$$T = \begin{pmatrix} \tau_{11} & \tau_{10} \\ \tau_{01} & \tau_{00} \end{pmatrix}.$$

It is not difficult to see that this T in fact coincides with our T defined in Section 2. By geometric consideration, it was shown in [1] that the imaginary part of T is indefinite and, thus, is not in the Siegel upper half-space. Instead, Klemm et al. defined

$$\mathrm{Z} = \overline{\mathrm{T}} + 2i\frac{\operatorname{Im}\mathrm{T}\, {}^t u u \operatorname{Im}\mathrm{T}}{u \operatorname{Im}\mathrm{T}\, {}^t u}, \quad u = (u_1\ u_0),$$

and showed that Z is in the Siegel upper half-space and transforms as

$$\mathrm{Z} \mapsto (A\mathrm{Z} + B)(C\mathrm{Z} + D)^{-1}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,$$

under the action of the monodromy group Γ. Therefore, if we consider z as a function of Z, then z behaves like a Siegel modular function. In this section, we will extend this result to general 4th order linear ordinary differential equations.

Theorem 2. *Let all the notation be given as in Section 2. Set*

$$\phi := (u_1\ u_0) \operatorname{Im}\mathrm{T} \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

and

$$\mathrm{Z} := \overline{\mathrm{T}} + 2i\phi^{-1} \operatorname{Im}\mathrm{T} \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} (u_1\ u_0) \operatorname{Im}\mathrm{T}.$$

Then, the action of a monodromy $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ *on* Z *is given by*

$$\gamma \colon \mathrm{Z} \mapsto \gamma\mathrm{Z} = (A\mathrm{Z} + B)(C\mathrm{Z} + D)^{-1}.$$

Also, if we consider z *and* $\phi\overline{w}_{01}$ *as functions of* Z*, then they satisfy*

$$z(\gamma\mathrm{Z}) = z(\mathrm{Z}), \qquad \phi(\gamma\mathrm{Z})\overline{w_{01}(\gamma\mathrm{Z})} = \det(C\mathrm{Z}+D)\phi(\mathrm{Z})\overline{w_{01}(\mathrm{Z})}.$$

Moreover, if the basis u_0, u_1, u_2, u_3 *satisfies*

(i) $\det(\operatorname{Im}\mathrm{T}) < 0$ (*i.e.,* $\operatorname{Im}\mathrm{T}$ *is indefinite*) *and*
(ii) $\operatorname{Im}(u_3\overline{u}_1 + u_2\overline{u}_0) > 0$,

then Z *is contained in the Siegel upper half-space.*

Remark 6. Note that properties (i) and (ii) assumed in the theorem are invariant under the action of the monodromy group $\Gamma \subset \mathrm{Sp}_4(\mathbb{R})$. The first one is very well-known. For the second property, we note that the inequality can be written as

$$\frac{1}{2i}\overline{u}J\,{}^t u > 0,$$

where $u = (u_3\ u_2\ u_1\ u_0)$ and $J = \left(\begin{smallmatrix} O & -E \\ E & O \end{smallmatrix}\right)$, O and E stand for the 2×2 zero and identity matrices, respectively. Now if $\gamma \in \Gamma$, then the action of γ gives

$$\frac{1}{2i}\overline{u}\,{}^t\gamma J \gamma\,{}^t u = \frac{1}{2i}\overline{u}J\,{}^t u > 0$$

since ${}^t\gamma J\gamma = J$.

Lemma 1. *Let all the notation be given as in Theorem* 2*. Then, under the action of* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$*, we have*

$$\phi(\gamma\mathrm{Z}) = (u_1\ u_0)\operatorname{Im}\mathrm{T}(C\overline{\mathrm{T}}+D)^{-1}(C\mathrm{T}+D)\begin{pmatrix} u_1 \\ u_0 \end{pmatrix}.$$

Moreover, we have the identity

$$\frac{\phi(\gamma\mathrm{Z})}{\phi(\mathrm{Z})} = \frac{\det(C\mathrm{Z}+D)}{\det(C\overline{\mathrm{T}}+D)}.$$

Proof. From (14), we have

$$\gamma : \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \mapsto (C\mathrm{T}+D)\begin{pmatrix} u_1 \\ u_0 \end{pmatrix}.$$

Also, a fundamental property of $\mathrm{Sp}_4(\mathbb{R})$ states that

$$\operatorname{Im}(A\mathrm{T}+B)(C\mathrm{T}+D)^{-1} = {}^t(C\mathrm{T}+D)^{-1}\operatorname{Im}\mathrm{T}(C\overline{\mathrm{T}}+D)^{-1}.$$

From these two properties, we see that

$$\gamma\colon \phi(\mathrm{Z}) \mapsto (u_1\ u_0)\,{}^t(C\mathrm{T}+D)\,{}^t(C\mathrm{T}+D)^{-1}\operatorname{Im}\mathrm{T}(C\overline{\mathrm{T}}+D)^{-1}(C\mathrm{T}+D)\begin{pmatrix} u_1 \\ u_0 \end{pmatrix}.$$

This yields the first identity in the lemma. The second identity can be verified by brute force. □

Proof of Theorem 2. The fact that Z transforms to $(A\mathrm{Z}+B)(C\mathrm{Z}+D)^{-1}$ can be verified by brute force, with the aid of the above lemma. The assertion that $z((A\mathrm{Z}+B)(C\mathrm{Z}+D)^{-1}) = z(\mathrm{Z})$ follows from the fact that z is invariant under the action of monodromy. We then combine Theorem 1 with the previous lemma to prove the claim about $\phi(\mathrm{Z})\overline{w_{01}(\mathrm{Z})}$. It remains to show that under the two conditions in the statements, the function Z is contained in the Siegel upper half-space.

By a direct computation, we find

$$\det(\operatorname{Im}\mathrm{Z}) = -\frac{\det(\operatorname{Im}\mathrm{T})}{|\phi(\mathrm{Z})|^2}\left((u_1\ u_0)\operatorname{Im}\mathrm{T}\begin{pmatrix}\overline{u}_1\\ \overline{u}_0\end{pmatrix}\right)^2.$$

Therefore, if $\operatorname{Im}\mathrm{T}$ is indefinite, then $\operatorname{Im}\mathrm{Z}$ is either positive definite or negative definite, depending on the sign of the $(1,1)$-entry of $\operatorname{Im}\mathrm{Z}$. Now if we write $\operatorname{Im}\mathrm{T} = \left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right)$, then the $(1,1)$-entry is

$$\left((u_1\ u_0)\operatorname{Im}\mathrm{T}\begin{pmatrix}\overline{u}_1\\ \overline{u}_0\end{pmatrix}\right)\left((u_1\ u_0)\begin{pmatrix}a^2 & ab\\ ab & 2b^2-ac\end{pmatrix}\begin{pmatrix}\overline{u}_1\\ \overline{u}_0\end{pmatrix}\right).$$

The second factor is a quadratic form of discriminant $4a^2(ac-b^2) = 4a^2\det(\operatorname{Im}\mathrm{T})$, which by assumption is negative. Thus the second factor is positive definite. Therefore, $\operatorname{Im}\mathrm{Z}$ is positive definite if

$$(u_1\ u_0)\operatorname{Im}\mathrm{T}\begin{pmatrix}\overline{u}_1\\ \overline{u}_0\end{pmatrix} > 0.$$

Using the relation

$$\mathrm{T}\begin{pmatrix}u_1\\ u_0\end{pmatrix} = \begin{pmatrix}u_3\\ u_2\end{pmatrix}$$

(cf. (11)), we find that the condition can be written as $\operatorname{Im}(u_3\overline{u}_1 + u_2\overline{u}_0) > 0$. This completes the proof of the theorem. □

4. Converse results

In Section 1, we have seen that if the projective monodromy group of a second order linear differential equation is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$, then one of the solutions of differential equations is a modular form of weight 1 under a suitable setting. Conversely, Theorem B shows that if we start out with a modular form u of weight 1 and a modular function z, then u as a function of z satisfies a second order linear differential equation. In this section we develop an analogous theory in the converse direction for Picard–Fuchs differential equations of order 4.

Given a fourth order Picard–Fuchs differential equation with symplectic monodromy Γ and a point of maximally unipotent monodromy, in Section 2 we have seen that if u_0, u_1, u_2, u_3 are solutions chosen in a way such that

$$w_{02} + w_{13} = 0, \qquad w_{jl} = W(u_j, u_l) = u_j\frac{\mathrm{d}u_l}{\mathrm{d}z} - u_l\frac{\mathrm{d}u_j}{\mathrm{d}z},$$

then

(1) setting $\tau_1 = w_{03}/w_{01}$, $\tau_2 = w_{02}/w_{01}$, $\tau_3 = -w_{12}/w_{01}$ and $\mathrm{T} = \left(\begin{smallmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{smallmatrix}\right)$, we have

$$\mathrm{T} \mapsto (A\mathrm{T}+B)(C\mathrm{T}+D)^{-1}$$

under the action of $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma$;

(2) w_{01} is Γ-modular form of weight 1 and z is a Γ-modular function in the sense that

$$z \mapsto z, \qquad w_{01} \mapsto \det(C\mathrm{T}+D)\cdot w_{01}$$

under the action of $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma$ (Theorem 1);

(3) $\mathrm{d}\tau_3/\mathrm{d}\tau_1 = (\mathrm{d}\tau_2/\mathrm{d}\tau_1)^2$ (identity (15)).

Here we show that if $w(\mathrm{T})$ is a Γ-modular form of weight 1 and $z(\mathrm{T})$ is a Γ-modular function in the sense of (1), (2) with $\mathrm{T} = \left(\begin{smallmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{smallmatrix}\right)$ satisfying property (3) above, then w as a function of z satisfies a fifth order linear differential equation (Theorems 3 and 4). Furthermore, there associates a fourth order linear differential equation whose projective monodromy group contains Γ (Theorems 5 and 6).

In order for our results to make sense, we shall impose the following assumptions.

Assumptions. Let Γ be a discrete subgroup of $\mathrm{Sp}_4(\mathbb{R})$ that may not be commensurable with $\mathrm{Sp}_4(\mathbb{Z})$. Let H be a connected domain in the upper half-plane $\mathbb{H}$. Assume that $t\colon H \to \mathbb{C}$ is a meromorphic function, and set, for $\tau \in H$,

$$\tau_1 = \tau, \qquad \tau_2 = -\int_{c_1}^{\tau_1} t(\tau)\,\mathrm{d}\tau, \quad \tau_3 = \int_{c_2}^{\tau_1} t(\tau)^2\,\mathrm{d}\tau,$$

and

$$\mathrm{T}(\tau) = \begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{pmatrix}.$$

Of course, τ_2 and τ_3 are multi-valued, depending on the choice of the paths of integration. Thus, we assume that for each residue r of $t(\tau)$, the matrix

$$\begin{pmatrix}1 & 0 & 0 & 2\pi ir\\ 0 & 1 & 2\pi ir & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

is contained in Γ, and so is

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 2\pi is\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

for each residue s of $t(\tau)^2$. Then let $\mathscr{C} \subset M_2(\mathbb{C})$ be the curve defined by

$$\mathscr{C} = \{\mathrm{T}(\tau) : \tau \in H\}$$

(with all possible branches of T included), and assume that the map $\Gamma\times\mathscr{C} \to M_2(\mathbb{C})$ given by

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\cdot \mathrm{T}(\tau) = (A\mathrm{T}(\tau)+B)(C\mathrm{T}(\tau)+D)^{-1} \tag{25}$$

defines a group action of Γ on $\mathscr{C}$.

Theorem 3. *Under the above assumptions, if $z, w\colon \mathscr{C} \to \mathbb{C}$ are non-constant meromorphic functions satisfying*

$$\begin{aligned} z(\gamma\mathrm{T}) &= z(\mathrm{T}),\\ w(\gamma\mathrm{T}) &= \chi(\gamma)\det(C\mathrm{T}+D)w(\mathrm{T})\end{aligned} \qquad \text{for all} \quad \gamma = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \Gamma, \tag{26}$$

where χ is a character of Γ, then w, $\tau_1 w$, $\tau_2 w$, $\tau_3 w$, and $(\tau_1\tau_3 - \tau_2^2)w$ viewed as functions of z are solutions of a fifth order linear differential equation whose coefficients are invariant under the substitution of T *by* $\gamma\mathrm{T}$ *for all* $\gamma \in \Gamma$.

Remark 7. Since the coefficients of Picard–Fuchs differential equations for Calabi–Yau threefolds are all rational functions, it is natural to conjecture that the quotient space $\Gamma\backslash\mathscr{C}$ in Theorem 3 can be compactified into a compact Riemann surface such

that (25) and (26) continue to hold on the compactified curve. We leave problems of this kind to future study.

The fifth order differential equation in Theorem 3 can be made more explicit. This is done in the next theorem.

Notation. Throughout this section, θ denotes $z\frac{\mathrm{d}}{\mathrm{d}z}$ while the prime $'$ stands for the differentiation with respect to τ.

Theorem 4. *Let* Γ, $t(\tau)$, $\mathrm{T}(\tau)$, $z(\mathrm{T})$, *and* $w(\mathrm{T})$ *be given as in Theorem* 3. *Define*

$$v=\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau},\quad G_1=\frac{\mathrm{d}z/\mathrm{d}\tau}{z},\quad G_2=\frac{\mathrm{d}w/\mathrm{d}\tau}{w},\quad G_3=\frac{\mathrm{d}v/\mathrm{d}\tau}{v}.$$

Then the fifth order differential equation in Theorem 3 *is*

$$\begin{aligned}
&\theta^5w+10p_1\theta^4w+(10\theta p_1+35p_1^2+5p_2)\theta^3w\\
&\quad+\Big(5\theta^2p_1+\frac{15}{2}\theta p_2+45p_1\theta p_1+50p_1^3+30p_1p_2\Big)\theta^2w\\
&\quad+\Big(46p_1^2\theta p_1+14p_2\theta p_1+24p_1^4+2p_3+4p_2^2+11p_1\theta^2p_1\\
&\qquad+\frac{9}{2}\theta^2p_2+\theta^3p_1+7(\theta p_1)^2+52p_1^2p_2+30p_1\theta p_2\Big)\theta w\\
&\quad+\big(4p_2\theta p_2+9p_1\theta^2p_2+7(\theta p_1)(\theta p_2)+26p_1^2\theta p_2+2p_2\theta^2p_1\\
&\qquad+20p_1p_2\theta p_1+\theta p_3+\theta^3p_2+4p_1p_3+24p_1^3p_2+8p_1p_2^2\big)w=0,
\end{aligned}$$

where

$$p_1=\frac{2G_1'-G_1(G_2+G_3)}{2G_1^2},\quad p_2=\frac{24G_3'-20(G_2+G_3)'+5(G_2+G_3)^2-8G_3^2}{20G_1^2},$$

and

$$p_3=\frac{-10G_3'''+40G_3G_3''+21(G_3')^2-54G_3^2G_3'+9G_3^4}{50G_1^4}$$

are functions invariant under the action of Γ.

Remark 8. Let $Lu=0$ be a Picard–Fuchs differential equation of the family of Calabi–Yau threefolds with symplectic monodromy. In practice, L has a singular point of maximally unipotent monodromy, which is usually assumed to be at $z=0$. Then the functions u_j set up at the beginning of this section can be chosen in a way such that u_0 is holomorphic at $z=0$, $u_1=c_1u_0\log z+\cdots$, $u_2=c_2u_0(\log z)^3+\cdots$, and $u_3=c_3u_0(\log z)^2+\cdots$. Then the Yukawa coupling K for the Calabi–Yau threefolds satisfies

$$K=C\frac{\mathrm{d}^2(u_3/u_0)}{\mathrm{d}(u_1/u_0)^2}=C\frac{u_0^3W(u_0,u_1,u_3)}{w_{01}^3}$$

for some constant C. In terms of τ_j given in (12), this can be expressed as

$$\frac{1}{K}=-\frac{1}{C}\,\frac{\mathrm{d}^2(w_{13}/w_{01})}{\mathrm{d}(w_{03}/w_{01})^2}=-\frac{1}{C}\,\frac{\mathrm{d}^2\tau_2}{\mathrm{d}\tau_1^2}.$$

Thus, when the fifth order differential equation in Theorem 4 arises from the antisymmetric square of L, the function v is actually equal to $-C/K$. Likewise, we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau_1}=\frac{\mathrm{d}}{\mathrm{d}(w_{03}/w_{01})}=\frac{C}{K}\,\frac{\mathrm{d}}{\mathrm{d}(u_1/u_0)}=-\frac{C}{K}\,\frac{\mathrm{d}}{\mathrm{d}t}.$$

In particular, we have

$$G_1 = \frac{C}{Kz}\frac{\mathrm{d}z}{\mathrm{d}(u_1/u_0)} = -\frac{C}{K}\frac{\mathrm{d}z/\mathrm{d}t}{z}.$$

Now let $f^{(k)}$ denote $\mathrm{d}^k f/\mathrm{d}(u_1/u_0)^k = (-1)^k \mathrm{d}^k f/\mathrm{d}t$. Then we have

$$G_1 = \frac{Cz^{(1)}}{Kz}, \qquad G_3 = -\frac{CK^{(1)}}{K^2}, \qquad G_3' = C^2\frac{2(K^{(1)})^2 - K^{(2)}K}{K^4},$$
$$G_3'' = C^3\frac{7K^{(1)}K^{(2)}K - 8(K^{(1)})^3 - K^2K^{(3)}}{K^6},$$

and

$$G_3''' = C^4\frac{11K^{(3)}K^{(1)}K^2 + 7K^2K^{(2)} - 59K^{(2)}(K^{(1)})^2K + 48(K^{(1)})^4 - K^{(4)}K^3}{K^8}.$$

Substituting these expressions into p_3 in Theorem 4 we find that the function

$$\frac{175(K^{(1)})^4 - 280K^{(2)}(K^{(1)})^2K + 49(K^{(2)}K)^2 + 70K^{(3)}K^{(1)}K^2 - 10K^{(4)}K^3}{K^4(z^{(1)})^4}$$

should be a function invariant under the action of monodromy. Indeed, it was noted in [11] and proved in [12] that the function above can be expressed in terms of the coefficients of the Picard–Fuchs differential equation.

Finally, the last piece of the converse theory shows that under the assumptions of Theorem 3, there does exist a fourth order linear differential equation whose projective monodromy group contains Γ.

Theorem 5. *Let all the notations be given as in Theorems* 3 *and* 4. *Write* $w_0 = w$, $w_j = \tau_j w$ *for* $j = 1, 2, 3$, *and* $w_4 = (\tau_1\tau_3 - \tau_2^2)w$. *Then the four functions*

$$\begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix}^{1/2}, \quad \begin{vmatrix} w_0 & \theta w_0 \\ w_3 & \theta w_3 \end{vmatrix}^{1/2}, \quad \begin{vmatrix} w_1 & \theta w_1 \\ w_4 & \theta w_4 \end{vmatrix}^{1/2}, \quad \begin{vmatrix} w_3 & \theta w_3 \\ w_4 & \theta w_4 \end{vmatrix}^{1/2}$$

viewed as functions of $z = z(\mathrm{T})$ *satisfy a fourth order linear differential equation whose coefficients are polynomials of* p_i *and their derivatives. Moreover, its projective monodromy group contains* Γ.

Remark 9. If we let $'$ denote the differentiation with respect to $\tau_1 = \tau$, the four functions in Theorem 5 can be alternatively expressed as

$$u = \begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix}^{1/2}, \quad \tau_2' u, \quad (\tau_1\tau_2' - \tau_2)u, \quad (\tau_2\tau_2' - \tau_3)u,$$

respectively, up to a sign. To see why this is so, we observe that

$$\frac{\begin{vmatrix} w_0 & \theta w_0 \\ w_3 & \theta w_3 \end{vmatrix}}{\begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix}} = \frac{\mathrm{d}(w_3/w_0)}{\mathrm{d}(w_1/w_0)} = \tau_3' = (\tau_2')^2.$$

This shows that

$$\begin{vmatrix} w_0 & \theta w_0 \\ w_3 & \theta w_3 \end{vmatrix}^{1/2} = \pm\tau_2' \begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix}^{1/2}.$$

The alternative expressions of the other two functions can be computed in the same way.

Remark 10. Note that one can never get the conclusion that the projective monodromy group equals to Γ in Theorem 5 because the functions z and w may actually be modular on a group Γ' strictly larger than what we assume. (Then the projective monodromy group will also contain Γ'.)

Fourth order pullbacks of the fifth order differential equation in Theorem 3 are by no means unique. In practice, we find that in most cases the following choice has simpler coefficients in the pullback differential equations.

Theorem 6. *Let all the notations be given as above, and let*

$$g = \exp\left\{-2\int^z p_1 \frac{\mathrm{d}z}{z}\right\}.$$

Then the four functions

$$g\begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix}^{1/2}, \quad g\begin{vmatrix} w_0 & \theta w_0 \\ w_3 & \theta w_3 \end{vmatrix}^{1/2}, \quad g\begin{vmatrix} w_1 & \theta w_1 \\ w_4 & \theta w_4 \end{vmatrix}^{1/2}, \quad g\begin{vmatrix} w_3 & \theta w_3 \\ w_4 & \theta w_4 \end{vmatrix}^{1/2}$$

satisfy the fourth order linear differential equation

$$\begin{aligned} &\theta^4 u + 16p_1\theta^3 u + \frac{1}{2}(187p_1^2 + 5p_2 + 38\theta p_1)\theta^2 u \\ &\quad + \frac{1}{2}\big(22\theta^2 p_1 + 5\theta p_2 + 294 p_1\theta p_1 + 472p_1^3 + 40p_1p_2\big)\theta u \\ &\quad + \frac{1}{16}\big(-8p_3 + 9p_2^2 + 12\theta^2 p_2 + 40\theta^3 p_1 + 160p_1\theta p_2 + 124p_2\theta p_1 \\ &\qquad + 4420p_1^2\theta p_1 + 680p_1\theta^2 p_1 + 460(\theta p_1)^2 + 622p_1^2p_2 + 3465p_1^4\big)u = 0. \end{aligned}$$

Remark 11. The reasons why a fourth order pullback exists in the form given in Theorem 5 and why the particular choice of pullbacks in Theorem 6 tends to have simpler coefficients can be explained as follows.

In general, given any four differentiable functions u_0, u_1, u_2, u_3 the wronskians have the relation

$$W(W(u_0,u_1), W(u_2,u_3)) = -u_0W(u_1,u_2,u_3) + u_1W(u_0,u_2,u_3).$$

This means that the antisymmetric square of the antisymmetric square of a linear differential equation is just the tensor product of the differential equation with its exterior cube (the differential equation satisfied by the wronskians $W(u_1,u_2,u_3)$ for any solutions u_j of the original differential equation). Now for a fourth order Picard–Fuchs differential equation $Lu = 0$ with symplectic monodromy, (18) shows that the exterior cube is essentially the same as L, except for an algebraic factor. Therefore, the antisymmetric square of the antisymmetric square of L is, up to an algebraic factor, the symmetric square of L. This explains the origin of the fourth order pullbacks. The reason why the pullback in Theorem 6 often has simpler coefficients is because it gets rid of the extra algebraic factor appearing in the exterior cube of L. See [2] for a more detailed computation and discussion.

The proof of these theorems will be given in the next section. Here we present some examples first.

Example 1. Recall that the modular group $\mathrm{SL}_2(\mathbb{R})$ can be naturally embedded in $\mathrm{Sp}_4(\mathbb{R})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A less obvious embedding is given by

$$\iota\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2d + 2abc & -3a^2c & abd + \frac{1}{2}b^2c & \frac{1}{2}b^2d \\ -a^2b & a^3 & -\frac{1}{2}ab^2 & -\frac{1}{6}b^3 \\ 4acd + 2bc^2 & -6ac^2 & ad^2 + 2bcd & bd^2 \\ 6c^2d & -6c^3 & 3cd^2 & d^3 \end{pmatrix}.$$

The origin of this embedding can be explained as follows.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $f(\tau)$ a modular form of weight 3 on Γ with character χ. Let $z(\tau)$ be a modular function on Γ. Then we have the equality of wronskians,

$$W(f, -\tfrac{1}{6}\tau^3 f) = f^2 W(1, -\tfrac{1}{6}\tau^3) = -f^2 \cdot \frac{1}{2}\tau^2 \frac{\mathrm{d}\tau}{\mathrm{d}z} = -f^2 W(\tau, \tfrac{1}{2}\tau^2) = -W(\tau f, \tfrac{1}{2}\tau^2 f),$$

hence if we let $\widehat{\gamma}$ denote the 4×4 matrix satisfying

$$\begin{pmatrix} \frac{1}{2}\tau^2 f \\ -\frac{1}{6}\tau^3 f \\ \tau f \\ f \end{pmatrix} \Big| \gamma = \widehat{\gamma} \cdot \begin{pmatrix} \frac{1}{2}\tau^2 f \\ -\frac{1}{6}\tau^3 f \\ \tau f \\ f \end{pmatrix} \qquad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

then, up to a numerical scalar, $\widehat{\gamma}$ is in the symplectic group. Indeed, a direct computation then shows that, up to the character χ, the matrix $\widehat{\gamma}$ is $\iota\gamma$.

Now let $t(\tau) = \tau$. Set

$$\tau_1 = \tau, \quad \tau_2 = -\int_0^\tau t(\tau)\,\mathrm{d}\tau = -\frac{1}{2}\tau^2, \quad \tau_3 = \int_0^\tau t(\tau)^2 \mathrm{d}\tau = \frac{1}{3}\tau^3$$

and

$$\mathrm{T}(\tau) = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \begin{pmatrix} \tau & -\frac{1}{2}\tau^2 \\ -\frac{1}{2}\tau^2 & \frac{1}{3}\tau^3 \end{pmatrix}. \tag{27}$$

(This is exactly the choice imposed by formulas (12) and (13).) For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, if we write $\iota\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ then we have

$$(A\mathrm{T} + B)(C\mathrm{T} + D)^{-1} = \begin{pmatrix} \gamma\tau & -\frac{1}{2}(\gamma\tau)^2 \\ -\frac{1}{2}(\gamma\tau)^2 & \frac{1}{3}(\gamma\tau)^3 \end{pmatrix} = \mathrm{T}(\gamma\tau),$$

where $\gamma\tau = (a\tau + b)/(c\tau + d)$. Thus, the mapping

$$(\iota\gamma, \mathrm{T}) \mapsto (A\mathrm{T} + B)(C\mathrm{T} + D)^{-1}$$

defines a group action of $\iota(\Gamma)$ on the set $\mathscr{C} = \{\mathrm{T}(\tau) : \tau \in \mathbb{H}\}$.

Moreover, we have

$$\det(C\mathrm{T} + D) = (c\tau + d)^4 (ad - bc) = (c\tau + d)^4.$$

Thus, if $w(\tau)$ is a modular form of weight 4 on Γ, then Theorem 3 implies that w, $\tau_1 w = \tau w$, $\tau_2 w = -\frac{1}{2}\tau^2 w$, $\tau_3 w = \frac{1}{3}\tau^3 w$, and $(\tau_1\tau_3 - \tau_2^2)w = \frac{1}{12}\tau^4 w$, as functions of z, satisfy a fifth order linear differential equation with algebraic functions as coefficients, in accordance with Theorem B.

Remark 12. Note that in the above example we have $\det \operatorname{Im} \mathrm{T}(\tau) = -(\operatorname{Im} \tau)^4/3$. Thus, the curve $\mathscr{C} = \{\mathrm{T}(\tau) : \tau \in \mathbb{H}\}$ is not contained in the Siegel upper half-space. In other words, the functions z and w in Theorem 3 may not be related to Siegel modular functions and modular forms at all.

Example 2. Consider the Picard–Fuchs differential equation (1) for the quintic threefolds. Let

$$y_0 = 1 + 120z + 113400z^2 + \cdots, \qquad y_1 = \frac{1}{2\pi i}(y_0 \log z + g_1),$$
$$y_2 = \frac{1}{(2\pi i)^2}\left(y_0 \frac{\log^2 z}{2} + g_1 \log z + g_2\right),$$
$$y_3 = \frac{1}{(2\pi i)^3}\left(y_0 \frac{\log^3 z}{6} + g_1 \frac{\log^2 z}{2} + g_2 \log z + g_3\right)$$

be the (normalized) Frobenius basis at $z = 0$. In [6] we showed that with respect to the ordered basis

$$u_3 = 5y_2 + \frac{5}{2}y_1 - \frac{25}{12}y_0, \quad u_2 = -5y_3 - \frac{25}{12}y_1 + \frac{200\zeta(3)}{(2\pi i)^3}y_0, \quad u_1 = y_1, \quad u_0 = y_0,$$

the monodromy matrices around $z = 0$ and $z = 1/3125$ are

$$\begin{pmatrix} 1 & 0 & 5 & 5 \\ -1 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

respectively. (Note that this is obtained from the collection (2) given in the introduction by reordering the basis.) Write $w_{jl} = W(u_j, u_l)$, $\tau_1 = w_{03}/w_{01}$, and $\tau_2 = w_{02}/w_{01}$. We find

$$w_{01} = \frac{1}{z} + 1010 + 1861650z + 4119140000z^2 + 9959217231250z^3 + \cdots,$$
$$\tau_1 = \frac{1}{2\pi i}\left(5\log z + 5\pi i + 6725z + \frac{16482625}{2}z^2 + \frac{44704818125}{3}z^3 + \cdots\right),$$

and

$$\tau_2 = -\frac{\tau_1^2}{10} + \frac{\tau_1}{2} + \frac{1}{(2\pi i)^2}\left(\frac{65}{6}\pi^2 + 2875z + \frac{17038125}{4}z^2 + \frac{151564765625}{18}z^3 + \cdots\right).$$

Then the functions v and G_j in Theorem 4 have the z-expansions

$$v = -\frac{1}{5} + 115z + 217500z^2 + 471493250z^3 + 1103069708750z^4 + \cdots$$

(notice that the Yukawa coupling has the z-expansion $5 + 2875z + 7090625z^2 + 18991003125z^3 + \cdots$, which is exactly $-1/v$),

$$G_1 = 2\pi i\left(\frac{1}{5} - 269z - 297500z^2 - 501290000z^3 - 1001288510000z^4 - \cdots\right),$$
$$G_2 = 2\pi i\left(-\frac{1}{5} + 471z + 566450z^2 + 1023038500z^3 + 2170808632500z^4 + \cdots\right),$$
$$G_3 = 2\pi i(-115z - 346450z^2 - 982613500z^3 - 2787375077500z^4 - \cdots).$$

We find

$$p_1 = \frac{1-6250z}{2(1-3125z)}, \qquad p_2 = \frac{1-17000z+37500000z^2}{(1-3125z)^2},$$
$$p_3 = \frac{5z(46+509375z+156250000z^2)}{(1-3125z)^4},$$

and the fifth order differential equation in Theorem 4 for the functions $\widetilde{w} = zw$ is

$$\theta^5\widetilde{w} - 5z(2\theta+1)(625\theta^4+1250\theta^3+1500\theta^2+875\theta+202)\widetilde{w}$$
$$+ 5^5z^2(5\theta+3)(5\theta+4)(5\theta+5)(5\theta+6)(5\theta+7)\widetilde{w} = 0.$$

(We normalize the functions $w \mapsto zw$ in order to have the local exponents zero with multiplicity 5 at $z=0$.) Using the `Maple` command `exterior_power`, we find that this is indeed the antisymmetric square of the differential equation (1).

Example 3. Consider the differential equation

$$\theta^5 y - 32z(2\theta+1)^5 y = 0 \tag{28}$$

with singularities at the points $z = 0$, $1/1024$, and ∞. Let y_0, y_1, y_2, y_3, y_4 denote its (normalized) Frobenius basis at $z=0$,

$$y_0 = f_0(z), \qquad y_1 = \frac{1}{2\pi i}\big(f_0(z)\log z + f_1(z)\big),$$
$$y_2 = \frac{1}{(2\pi i)^2}\left(f_0(z)\frac{\log^2 z}{2} + f_1(z)\log z + f_2(z)\right),$$
$$y_3 = \frac{1}{(2\pi i)^3}\left(f_0(z)\frac{\log^3 z}{3!} + f_1(z)\frac{\log^2 z}{2} + f_2(z)\log z + f_3(z)\right),$$
$$y_4 = \frac{1}{(2\pi i)^4}\left(f_0(z)\frac{\log^4 z}{4!} + f_1(z)\frac{\log^3 z}{3!} + f_2(z)\frac{\log^2 z}{2} + f_3(z)\log z + f_4(z)\right).$$

Following Beukers' argument [3, Sections 3 and 4], we can show that these functions satisfy

$$y_0y_4 - y_1y_3 + \frac{1}{2}y_2^2 = 0, \qquad (\theta y_0)(\theta y_4) - (\theta y_1)(\theta y_3) + \frac{1}{2}(\theta y_2)^2 = 0$$

Thus, up to conjugation, the monodromy group is contained in the orthogonal group O_5. Using the method in [6] we can prove that, relative to the ordered basis y_4, y_3, y_2, y_1, y_0, the monodromy matrices around $z=0$ and $z=1/1024$ are

$$\begin{pmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 \\ 0 & 1 & 1 & 1/2 & 1/6 \\ 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} a^2 & 0 & -ab & (1-a^2)x & -b^2/2 \\ -c^2x/2 & 1 & -acx & c^2x^2/2 & -(1-a^2)x \\ -ac & 0 & 1-2a^2 & acx & -ab \\ 0 & 0 & 0 & 1 & 0 \\ -c^2/2 & 0 & -ac & c^2x/2 & a^2 \end{pmatrix},$$

where $a = 5/6$, $b = 11/144$, $c = 8$, and $x = 10\zeta(3)/(2\pi i)^3$ (see [6, Theorem 3]). Set

$$\begin{pmatrix} w_4 \\ w_3 \\ w_2 \\ w_1 \\ w_0 \end{pmatrix} = \begin{pmatrix} 32 & 0 & 20/3 & -32x & -25/36 \\ 0 & 8 & 0 & 0 & -8x \\ 0 & 0 & -4 & 0 & 5/6 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \\ y_0 \end{pmatrix}.$$

With respect to this new basis, the matrices become

$$\begin{pmatrix} 1 & 4 & -4 & 3 & 8 \\ 0 & 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also, w_j satisfy

$$w_0w_4 - w_1w_3 + w_2^2 = 0, \qquad (\theta w_0)(\theta w_4) - (\theta w_1)(\theta w_3) + (\theta w_2)^2 = 0.$$

Now for $j = 1, \dots, 4$, let $\tau_j = w_j/w_0$, and let τ_j' denote the derivative of τ_j with respect to $\tau = \tau_1$. From the above relations we deduce that

$$\tau_4 = \tau_1\tau_3 - \tau_2^2, \qquad \tau_3' = (\tau_2')^2.$$

Set $\mathrm{T}(\tau) = \mathrm{T}(\tau_1) = \left(\begin{smallmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{smallmatrix}\right)$. Around $z = 0$, we have $w_0 \mapsto w_0$ and

$$\begin{aligned} \mathrm{T}(\tau) &\mapsto \begin{pmatrix} \tau_1 + 4 & \tau_2 - \tau_1 - 2 \\ \tau_2 - \tau_1 - 2 & \tau_3 - 2\tau_2 + \tau_1 + 3 \end{pmatrix} \\ &= \left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathrm{T}(\tau) + \begin{pmatrix} 4 & 2 \\ -2 & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathrm{T}(\tau) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)^{-1}. \end{aligned}$$

Around $z = 1/1024$, we have

$$w_0 \mapsto -\tau_4 w_0 = -(\tau_1\tau_3 - \tau_2^2)w_0 = -\det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{T}(\tau) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) w_0$$

and

$$\begin{aligned} \mathrm{T}(\tau) &\mapsto -\frac{1}{\tau_4} \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathrm{T}(\tau) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{T}(\tau) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)^{-1}. \end{aligned}$$

Thus, letting Γ be the subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ generated by

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 4 & 2 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \gamma_1^{-1}, \tag{29}$$

$w_0(\mathrm{T})$ is a Γ-modular form of weight 1. The functions v and G_j in Theorem 4 have the z-expansions

$$\begin{aligned} v &= 1 + 160z + 132320z^2 + 115614720z^3 + 104797147360z^4 + \cdots, \\ G_1 &= 2\pi i(1 - 160z - 54880z^2 - 29946880z^3 - 19691390560z^4 - \cdots), \\ G_2 &= 2\pi i(32z + 9408z^2 + 4805632z^3 + 3045669248z^4 + \cdots), \\ G_3 &= 2\pi i(160z + 213440z^2 + 240399360z^3 + 259173946240z^4 + \cdots). \end{aligned}$$

We find

$$p_1 = \frac{-256z}{1 - 1024z}, \quad p_2 = \frac{65536z^2}{(1 - 1024z)^2}, \quad p_3 = \frac{-32z - 163840z^2 - 33554432z^3}{(1 - 1024z)^4}.$$

Of course, the fifth order differential equation in Theorem 4 is just the original hypergeometric differential equation, while the fourth order differential equation in Theorem 6 is

$$\theta^4 - 16z(128\theta^4 + 256\theta^3 + 304\theta^2 + 176\theta + 39) + 2^{20}z^2(\theta+1)^4. \tag{30}$$

Example 3 is our basic example for further illustrations. In Section 6 we discuss some arithmetic observations around this example, while Theorem VP1 in the Appendix shows that Γ is not of finite index in $\mathrm{Sp}_4(\mathbb{Z})$.

5. Proof of the converse theorems

Notation. For

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \Gamma \subset \mathrm{Sp}_4(\mathbb{R})$$

and $\tau \in H$ we write

$$C\mathrm{T} + D = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} = \begin{pmatrix} a_{31}\tau_1 + a_{32}\tau_2 + a_{33} & a_{31}\tau_2 + a_{32}\tau_3 + a_{34} \\ a_{41}\tau_1 + a_{42}\tau_2 + a_{43} & a_{41}\tau_2 + a_{42}\tau_3 + a_{44} \end{pmatrix}. \tag{31}$$

The notation $\gamma\tau$ will represent the $(1,1)$-entry of $\gamma\mathrm{T}(\tau) = (A\mathrm{T}+B)(C\mathrm{T}+D)^{-1}$. More generally, for a function $g(\tau)$ of τ we often write $\gamma g(\tau)$ or $g|\gamma$ in place of $g(\gamma\tau)$.

For ease of notation, we let M^{adj} denote the adjugate of a square matrix M, i.e., M^{adj} is the square matrix such that $M^{\mathrm{adj}}M = (\det M)\,\mathrm{Id}$.

We start out by doing some elementary computation.

Lemma 2. *Let* $\mathrm{T} = \mathrm{T}(\tau)$ *be defined as in Theorem* 3*. For* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$*, we have*

$$\mathrm{d}(\gamma\mathrm{T}) = \frac{{}^t(C\mathrm{T}+D)^{\mathrm{adj}}\,\mathrm{d}\mathrm{T}\,(C\mathrm{T}+D)^{\mathrm{adj}}}{\det(C\mathrm{T}+D)^2}, \tag{32}$$

$$\frac{\mathrm{d}\gamma\tau}{\mathrm{d}\tau} = \frac{(\mathfrak{c}t+\mathfrak{d})^2}{\det(C\mathrm{T}+D)^2}, \tag{33}$$

$$\frac{\mathrm{d}\mathrm{T}(\tau)}{\mathrm{d}\tau}|\gamma = \frac{{}^t(C\mathrm{T}+D)^{\mathrm{adj}}\,(\mathrm{d}\mathrm{T}/\mathrm{d}\tau)\,(C\mathrm{T}+D)^{\mathrm{adj}}}{(\mathfrak{c}t+\mathfrak{d})^2}, \tag{34}$$

and

$$t(\gamma\tau) = \frac{\mathfrak{a}t(\tau)+\mathfrak{b}}{\mathfrak{c}t(\tau)+\mathfrak{d}} \text{ for all } \gamma \in \Gamma \text{ and } \tau \in \mathbb{H}. \tag{35}$$

Proof. Identity (32) is simply a restatement of the basic property

$$\mathrm{d}(A\mathrm{T}+B)(C\mathrm{T}+D)^{-1} = {}^t(C\mathrm{T}+D)^{-1}\,\mathrm{d}\mathrm{T}\,(C\mathrm{T}+D)^{-1}$$

of the symplectic group.

To prove (33), we compare the $(1,1)$-entries of the two sides of (32). The $(1,1)$-entry of the left-hand side is $\mathrm{d}\gamma\tau/\mathrm{d}\tau$, while the numerator of the right-hand side

is

$$
\begin{aligned}
(36)\qquad {}^t(C\mathrm{T}+D)^{\mathrm{adj}}\frac{\mathrm{dT}(\tau)}{\mathrm{d}\tau}(C\mathrm{T}+D)^{\mathrm{adj}} &= \begin{pmatrix} \mathfrak{d} & -\mathfrak{c} \\ -\mathfrak{b} & \mathfrak{a} \end{pmatrix}\begin{pmatrix} 1 & -t \\ -t & t^2 \end{pmatrix}\begin{pmatrix} \mathfrak{d} & -\mathfrak{b} \\ -\mathfrak{c} & \mathfrak{a} \end{pmatrix} \\
&= \begin{pmatrix} (\mathfrak{c}t+\mathfrak{d})^2 & -(\mathfrak{a}t+b)(\mathfrak{c}t+\mathfrak{d}) \\ -(\mathfrak{a}t+b)(\mathfrak{c}t+\mathfrak{d}) & (\mathfrak{a}t+\mathfrak{b})^2 \end{pmatrix},
\end{aligned}
$$

whose $(1,1)$-entry is $(\mathfrak{c}t+\mathfrak{d})^2$. This proves identity (33).

Finally, (34) follows from the first two, and (35) follows from (34) and (36). This completes the proof. $\square$

Lemma 3. *Let $v(\tau)=\mathrm{d}t/\mathrm{d}\tau$. Then*

$$
(37)\qquad v(\gamma\tau)=\frac{\det(C\mathrm{T}+D)^3}{(\mathfrak{c}t+\mathfrak{d})^4}v(\tau).
$$

Proof. Differentiating (35) we obtain

$$
v(\gamma\tau)\frac{\mathrm{d}\gamma\tau}{\mathrm{d}\tau}=\frac{(\mathfrak{a}'t+\mathfrak{b}')(\mathfrak{c}t+\mathfrak{d})-(\mathfrak{a}t+\mathfrak{b})(\mathfrak{c}'t+\mathfrak{d}')}{(\mathfrak{c}t+\mathfrak{d})^2}+\frac{\mathfrak{a}\mathfrak{d}-\mathfrak{b}\mathfrak{c}}{(\mathfrak{c}t+\mathfrak{d})^2}v(\tau).
$$

We then observe that from (31) we have $\mathfrak{a}\mathfrak{d}-\mathfrak{b}\mathfrak{c}=\det(C\mathrm{T}+D)$ and $\mathfrak{a}'t+\mathfrak{b}'=\mathfrak{c}'t+\mathfrak{d}'=0$. Then from (33) in Lemma 2 we obtain (37). $\square$

Lemma 4. *Let G_j and p_j, $j=1,2,3$, be defined as in Theorem 4. Then $p_j(\gamma\tau)=p_j(\tau)$ for all $\gamma\in\Gamma$.*

Proof. For the sake of convenience, for $\gamma=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\in\Gamma$ and $\mathrm{T}=\mathrm{T}(\tau)\in\mathscr{C}$, set

$$
g(\gamma,\mathrm{T})=\mathfrak{c}t+\mathfrak{d}, \qquad h(\gamma,\mathrm{T})=\det(C\mathrm{T}+D).
$$

Taking the logarithmic derivatives of the two sides of (26) and (37) with respect to τ and then using (33), we obtain

$$
(38)\qquad G_1(\gamma\tau)=\frac{h^2}{g^2}G_1(\tau),
$$

$$
(39)\qquad G_2(\gamma\tau)=\frac{h'h}{g^2}+\frac{h^2}{g^2}G_2(\tau),
$$

and

$$
(40)\qquad G_3(\gamma\tau)=3\frac{h'h}{g^2}-4\frac{g'h^2}{g^3}+\frac{h^2}{g^2}G_3(\tau).
$$

Differentiating (38) with respect to τ_1 again, we have

$$
G_1'(\gamma\tau)\frac{\mathrm{d}\gamma\tau}{\mathrm{d}\tau}=2\frac{h'h}{g^2}G_1(\tau)-2\frac{g'h^2}{g^3}G_1(\tau)+\frac{h^2}{g^2}G_1'(\tau).
$$

From this, (33), (39), and (40) we deduce that

$$
\left(G_1'-\frac{G_1(G_2+G_3)}{2}\right)\Big|\gamma=\frac{h^4}{g^4}\left(G_1'-\frac{G_1(G_2+G_3)}{2}\right).
$$

It follows that $p_1=(G_1'-G_1(G_2+G_3)/2)/G_1^2$ is invariant under the action of Γ.

We next prove that p_2 is invariant under Γ. By a direct computation, we find

$$\begin{aligned}
&\bigl(24G_3' - 20(G_2+G_3)' + 5(G_2+G_3)^2 - 8G_3^2\bigr)|\gamma \\
&\qquad - \frac{h^4}{g^4}\bigl(24G_3' - 20(G_2+G_3)' + 5(G_2+G_3)^2 - 8G_3^2\bigr) \\
&\quad = 8\frac{h^3}{g^5}(2g'h' + 2g'hG_3 - 2g''h - h''g) \\
&\quad = 8\frac{h^3}{g^5}(a_{41}a_{33} + a_{42}a_{34} - a_{31}a_{43} - a_{32}a_{44})gv.
\end{aligned}$$

The property $C\,{}^tD = D\,{}^tC$ of the symplectic matrix γ implies $a_{41}a_{33} + a_{42}a_{34} - a_{31}a_{43} - a_{32}a_{44} = 0$. It follows that p_2 is invariant under the action of Γ.

The invariance of p_3 under Γ can be proved in the same way. We repeatedly use $2g'h' + 2g'hG_3 - 2g''h - h''g = 0$ just shown and (33). The details are too complicated to be presented here. $\square$

Proof of Theorems 3 *and* 4. We first give a conceptual proof of Theorem 3, which may be of interest independent of our later proof of Theorem 4.

For convenience, we let $\mathbf{w}$ denote the column vector ${}^t((\tau_1\tau_3-\tau_2^2)w, \tau_3 w, \tau_2 w, \tau_1 w, w)$. Given $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma$, we have

$$\begin{aligned}
\bigl(w(\mathrm{T})\mathrm{T}\bigr)|\gamma &= \chi(\gamma)\det(C\mathrm{T}+D)w(\mathrm{T})(A\mathrm{T}+B)(C\mathrm{T}+D)^{-1} \\
&= \chi(\gamma)w(\mathrm{T})(A\mathrm{T}+B)(C\mathrm{T}+D)^{\mathrm{adj}} \\
&= \chi(\gamma)w(\mathrm{T})\bigl((\det\mathrm{T})AC^{\mathrm{adj}} + A\mathrm{T}D^{\mathrm{adj}} + B\mathrm{T}^{\mathrm{adj}}C^{\mathrm{adj}} + BD^{\mathrm{adj}}\bigr).
\end{aligned}$$

This shows that there is a matrix $\widehat{\gamma}$ in $M_5(\mathbb{R})$ such that

$$\mathbf{w}|\gamma = \chi(\gamma)\widehat{\gamma}\cdot\mathbf{w}.$$

Since $z(\mathrm{T})$ is assumed to be invariant under the substitution $\mathrm{T} \mapsto \gamma\mathrm{T}$, the same matrix $\widehat{\gamma}$ also satisfies

$$\theta^j\mathbf{w}|\gamma = \chi(\gamma)\widehat{\gamma}\cdot\theta^j\mathbf{w} \qquad \text{for} \quad j = 1, 2, \dots .$$

Therefore, the coefficients $r_j(\mathrm{T})$ in the linear dependence

$$\theta^5\mathbf{w} = \sum_{j=0}^{4} r_j(\mathrm{T})\theta^j\mathbf{w}$$

must be invariant under the action of γ. This proves Theorem 3. We now prove Theorem 4.

Setting $G_2/G_1 = \lambda$ we have

$$\theta w = t\frac{\mathrm{d}w/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau} = w\lambda \tag{41}$$

and

$$\begin{aligned}
\theta\lambda &= \frac{1}{G_1}\left(\frac{G_2}{G_1}\right)' \\
&= \frac{20G_2' - 4G_3' - 5G_2^2 - 10G_2G_3 + 3G_3^2}{20G_1^2} \\
&\qquad - \frac{G_2(2G_1' - G_1(G_2+G_3))}{2G_1^3} - \frac{G_2^2}{4G_1^2} + \frac{4G_3' - 3G_3^2}{20G_1^2} \\
&= -p_2 - p_1\lambda - \frac{1}{4}\lambda^2 + \mu,
\end{aligned} \tag{42}$$

where we let

$$p_1 = \frac{2G_1' - G_1(G_2+G_3)}{G_1^2}, \quad p_2 = \frac{4G_3' - 20G_2' + 5G_2^2 + 10G_2G_3 - 3G_3^2}{G_1^2}, \tag{43}$$

and

$$\mu = \frac{4G_3' - 3G_3^2}{20G_1^2}.$$

Note that p_1 and p_2 are invariant under the action of Γ by Lemma 4.

Furthermore, by a similar argument we have

$$\theta\mu = \frac{1}{G_1}\left(\frac{4G_3' - 3G_3^2}{20G_1^2}\right) = -2p_1\mu - \lambda\mu + \nu, \tag{44}$$

where

$$\nu = \frac{4G_3'' - 10G_3'G_3 + 3G_3^3}{20G_1^3};$$

then

$$\theta\nu = -3p_1\nu - \frac{3}{2}\lambda\nu - 2\mu^2 - p_3, \tag{45}$$

where

$$p_3 = \frac{-10G_3''' + 40G_3G_3'' + 21(G_3')^2 - 54G_3^2G_3' + 9G_3^4}{50G_1^4}$$

is a function invariant under Γ by Lemma 4. Using (41)–(45) we find

$$\begin{aligned}
\theta^2 w &= \left(\frac{3}{4}\lambda^2 - p_1\lambda - p_2 + \mu\right)w, \\
\theta^3 w &= \left(\frac{3}{8}\lambda^3 - \frac{9}{4}p_1\lambda^2 + \left(p_1^2 - \theta p_1 - \frac{5}{2}p_2 + \frac{3}{2}\mu\right)\lambda \right. \\
&\qquad \left. - \theta p_2 + p_1p_2 - 3p_1\mu + \nu\right)w, \\
\theta^4 w &= \left(\frac{3}{32}\lambda^4 - \frac{9}{4}p_1\lambda^3 + \left(\frac{21}{4}p_1^2 - 3\theta p_1 - 3p_2 + \frac{3}{4}\mu\right)\lambda^2 + \cdots\right)w,
\end{aligned}$$

and

$$\theta^5 w = \left(-\frac{15}{16}p_1\lambda^4 + \left(\frac{75}{8}p_1^2 - \frac{15}{4}\theta p_1 - \frac{15}{8}p_2\right)\lambda^3 + \cdots\right)w.$$

It follows that

$$\begin{aligned}\theta^5 w &= -10p_1\theta^4 w + \left(-\frac{1}{8}(105p_1^2 + 30\theta p_1 + 15p_2)\lambda^3 + \cdots\right)w \\ &= -10p_1\theta^4 w - (35p_1^2 + 10\theta p_1 + 5p_2)\theta^3 w \\ &\quad + \left(-\frac{1}{8}(300p_1^3 + 270p_1\theta p_1 + 30\theta^2 p_1 + 180p_1p_2 + 45\theta p_2)\lambda^2 + \cdots\right)w \\ &= -10p_1\theta^4 w - (35p_1^2 + 10\theta p_1 + 5p_2)\theta^3 w \\ &\quad - \left(50p_1^3 + 45p_1\theta p_1 + 5\theta^2 p_1 + 30p_1p_2 + \frac{15}{2}\theta p_2\right)\theta^2 w + \cdots .\end{aligned}$$

Continuing this way, we find that w satisfies the differential equation given in the statement of Theorem 4. $\square$

Proof of Theorems 5 *and* 6. Here we shall adopt all the notations in the proof of Theorems 3 and 4. In particular, we set

$$\lambda = \frac{G_2}{G_1}, \qquad \mu = \frac{4G_3' - 3G_3^2}{20G_1^2}, \qquad \nu = \frac{4G_3'' - 10G_3'G_3 + 3G_3^3}{20G_1^3}.$$

We first give a proof of Theorem 6.

Observe that

$$\begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix} = w_0^2 z \frac{\mathrm{d}\tau_1}{\mathrm{d}z} = \frac{w_0^2}{G_1}. \tag{46}$$

Setting $u = g\left|\begin{smallmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{smallmatrix}\right|^{1/2}$, we have, by (41), (43) and $\theta = z\mathrm{d}/\mathrm{d}z = G_1^{-1}\mathrm{d}/\mathrm{d}\tau$,

$$\theta u = u\left(\lambda - \frac{G_1'}{G_1^2} - 2p_1\right) = u\left(\frac{3}{4}\lambda - \frac{1}{4}\rho + \frac{5}{2}p_1\right),$$

where we set

$$\rho = \frac{G_3}{G_1}.$$

Then, by (43),

$$\theta\rho = 5\mu - p_1\rho - \frac{1}{2}\lambda\rho + \frac{1}{4}\rho^2. \tag{47}$$

Using (42), (44), (45) we find

$$\begin{aligned}\theta^2 u &= u\biggl(\frac{3}{8}\lambda^2 - \left(\frac{1}{4}\rho + \frac{9}{2}p_1\right)\lambda - \frac{1}{4}(10\theta p_1 + 3p_2 - 25p_1^2 + 2\mu - 6p_1\rho)\biggr), \\ \theta^3 u &= u\biggl(\frac{3}{32}\lambda^3 - \left(\frac{3}{32}\rho + \frac{63}{16}p_1\right)\lambda^2 \\ &\qquad - \frac{3}{16}\left(34\theta p_1 + 7p_2 - 109p_2^2 + 2\mu - 14p_1\rho\right)\lambda + \cdots\biggr),\end{aligned}$$

and

$$\theta^4 u = u\biggl(-\frac{3}{2}p_1\lambda^3 - \frac{3}{16}\left(38\theta p_1 + 5p_2 - 149p_1^2 - 8p_1\rho\right)\lambda^2 + \cdots\biggr).$$

From these computations, we see that $g\left|\begin{smallmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{smallmatrix}\right|^{1/2}$ satisfies the differential equation in Theorem 6. By a similar argument, we can show that the other three functions also satisfy the same differential equation.

We now prove the monodromy part of Theorem 5. Set

$$u_0 = \begin{vmatrix} w_0 & \theta w_0 \\ w_1 & \theta w_1 \end{vmatrix}^{1/2}, \quad u_1 = -\tau_2' u_0, \quad u_2 = (\tau_3 - \tau_2\tau_2')u_0, \quad u_3 = (\tau_2 - \tau_1\tau_2')u_0.$$

According to Remark 9, up to a sign, these functions are the same as the four functions given in the statement of Theorem 5. Let

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \Gamma.$$

It suffices to show that under the action of γ,

$$\begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{pmatrix} \mapsto \epsilon\gamma \begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{pmatrix} \tag{48}$$

for some complex scalar ϵ depending on γ.

Using (46) and (38) we find that under the action of γ,

$$u_0^2 = \frac{w_0^2}{G_1} \mapsto \chi(\gamma)^2(\mathfrak{c}t+\mathfrak{d})^2\frac{w_0^2}{G_1}.$$

Recalling that $t = -\tau_2'$ by our formulation we have

$$\mathfrak{c}t + \mathfrak{d} = a_{41}(-\tau_1\tau_2 + \tau_2) + a_{42}(-\tau_2\tau_2 + \tau_3) - a_{43}\tau_2' + a_{44}.$$

It follows that

$$u_0 \mapsto \epsilon(a_{41}u_3 + a_{42}u_2 + a_{43}u_1 + a_{44}u_0)$$

under the action of γ for some scalar ϵ.

For the behavior of u_1 under γ, we use (35). We find

$$u_1 = tu_0 \mapsto \frac{\mathfrak{a}t+\mathfrak{b}}{\mathfrak{c}t+\mathfrak{d}}\left(\epsilon(\mathfrak{c}t+\mathfrak{d})u_0\right) = \epsilon(a_{31}u_3 + a_{32}u_2 + a_{33}u_1 + a_{34}u_0).$$

For u_2 and u_3, we have

$$\begin{pmatrix} u_3 \\ u_2 \end{pmatrix} = \mathrm{T}\begin{pmatrix} t \\ 1 \end{pmatrix} u_0.$$

Under the action of γ we have $\mathrm{T} \mapsto (A\mathrm{T}+B)(C\mathrm{T}+D)^{-1}$ and by (35)

$$\begin{pmatrix} t \\ 1 \end{pmatrix} u_0 \mapsto \epsilon \begin{pmatrix} \mathfrak{a}t+\mathfrak{b} \\ \mathfrak{c}t+\mathfrak{d} \end{pmatrix} u_0 = \epsilon(C\mathrm{T}+D)\begin{pmatrix} t \\ 1 \end{pmatrix} u_0.$$

It follows that

$$\begin{pmatrix} u_3 \\ u_2 \end{pmatrix} \mapsto \epsilon(A\mathrm{T}+B)\begin{pmatrix} t \\ 1 \end{pmatrix} u_0$$

under the action of γ. From this we deduce that

$$u_2 \mapsto \epsilon(a_{21}u_3 + a_{22}u_2 + a_{23}u_1 + a_{24}u_0), \quad u_3 \mapsto \epsilon(a_{11}u_3 + a_{12}u_2 + a_{13}u_1 + a_{14}u_0).$$

This establishes (48) and completes the proof of the theorems. □

6. Guillera's generalization of Ramanujan's formulas for $1/\pi$

The modular parameterization of solutions of Picard–Fuchs linear differential equations of order 3 has another curious application to proving Ramanujan's series for $1/\pi$ [14], like

$$\sum_{n=0}^{\infty} \frac{(4n)!}{n!^4}(26390n+1103)\cdot\frac{1}{396^{4n}} = \frac{99^2}{2\pi\sqrt{2}}.$$

Note that the series on the left-hand side is a $\mathbb{Q}$-linear combination of the ${}_3F_2$ hypergeometric series satisfying (3) and its derivative at a point close to the origin. The paper [18] reviews ideas of proofs of Ramanujan's series and its several generalizations.

As already mentioned in Section 1, modular Picard–Fuchs differential equations of order 3 always come as the symmetric square of equations of order 2. We consider such a second order differential equation and proceed in the notation of Section 1 until equality (5), where we choose $C = 2\pi i$.

From (6) we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = \frac{W(u_0,u_1)}{u_0^2} = \frac{1}{2\pi i g_0 u_0^2},$$

hence

$$\delta = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{2\pi i}\left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}z} = g_0u_0^2\frac{\mathrm{d}}{\mathrm{d}z}. \tag{49}$$

Our next object is the function

$$v = v(\tau) = \delta \log u_0. \tag{50}$$

From (49) we obtain

$$v = \frac{\delta u_0}{u_0} = g_0(z)u_0u_0'. \tag{51}$$

Lemma 5. *The following functional equation is valid for any $\gamma \in \Gamma$:*

$$v(\gamma\tau) = (c\tau+d)^2v(\tau) + \frac{1}{2\pi i}c(c\tau+d). \tag{52}$$

Proof. Indeed,

$$\begin{aligned} v(\gamma\tau) &= \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}(\gamma\tau)}\log(cu_1+du_0) \\ &= \left(\frac{\mathrm{d}(\gamma\tau)}{\mathrm{d}\tau}\right)^{-1}\cdot\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\big(u_0\cdot(c\tau+d)\big) \\ &= (c\tau+d)^2\cdot\left(\delta\log u_0 + \frac{1}{2\pi i}\frac{c}{c\tau+d}\right) \\ &= (c\tau+d)^2v(\tau) + \frac{1}{2\pi i}c(c\tau+d). \end{aligned}$$

Lemma 6. *For any integer $N \ge 2$, the function $\widetilde{v}(\tau) = v(\tau) - Nv(N\tau)$ is a Γ'-modular form of weight 2, where*

$$\Gamma' = \Gamma'_N = \left\{\gamma = \begin{pmatrix} a & b \\ cN & d\end{pmatrix} \in \Gamma : \gamma^* = \begin{pmatrix} a & bN \\ c & d\end{pmatrix} \in \Gamma\right\}.$$

A consequence of this lemma is that $\widetilde{v} = g_1(z)u_0^2$, where g_1 is an algebraic function of z.

Proof. For any $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma'$ we have

$$N \cdot \gamma\tau = N \cdot \frac{a\tau + b}{cN\tau + d} = \gamma^*(N\tau),$$

hence, by (52),

$$\begin{aligned}
\widetilde{v}(\gamma\tau) &= v(\gamma\tau) - Nv(N\cdot\gamma\tau) = v(\gamma\tau) - Nv(\gamma^*(N\tau)) \\
&= (cN\cdot\tau + d)^2 v(\tau) + \frac{1}{2\pi i} cN(cN\cdot\tau + d) \\
&\quad - N\biggl((c\cdot N\tau + d)^2 v(N\tau) + \frac{1}{2\pi i} c(c\cdot N\tau + d)\biggr) \\
&= (cN\tau + d)^2\bigl(v(\tau) - Nv(N\tau)\bigr) = (cN\tau + d)^2\widetilde{v}(\tau)
\end{aligned}$$

that implies the desired assertion.

It is now time to glue the gathered information. Take a quadratic irrationality τ_0 (from the upper half-plane) and an element $\gamma_0 \in \Gamma$ such that

$$\gamma_0\tau_0 = N\tau_0 \quad \text{for some integer } N \ge 2. \tag{53}$$

Lemma 7. *In the above notation, the number $z_0 = z(\tau_0) = z(\gamma_0\tau_0)$ is algebraic.*

Proof. This follows from the fact that $z(\tau)$ and $z(N\tau)$ are connected by a (modular) polynomial equation with integer coefficients. Substituting $\tau = \tau_0$ into this equation gives a polynomial for the number $z(\tau_0) = z(N\tau_0)$.

From $v(\tau_0) - Nv(N\tau_0) = \widetilde{v}(\tau_0) = g_1(z_0)u_0(z_0)^2$ and Lemma 5 it follows that

$$\begin{aligned}
v(N\tau_0) = v(\gamma_0\tau_0) &= (c_0\tau_0 + d_0)^2 v(\tau_0) + \frac{1}{2\pi i} c_0(c_0\tau_0 + d_0) \\
&= (c_0\tau_0 + d_0)^2\bigl(Nv(N\tau_0) + g_1(z_0)u_0(z_0)^2\bigr) + \frac{1}{2\pi i} c_0(c_0\tau_0 + d_0),
\end{aligned}$$

hence

$$v(N\tau_0) = \frac{(c_0\tau_0 + d_0)^2 g_1(z_0)u_0(z_0)^2 + c_0(c_0\tau_0 + d_0)/2\pi i}{1 - N\cdot(c_0\tau_0 + d_0)^2}. \tag{54}$$

On the other hand, from (51) we have

$$v(N\tau_0) = \frac{1}{2} g_0(z)\bigl(u_0(z)^2\bigr)'\big|_{z=z_0}. \tag{55}$$

It remains to eliminate $v(N\tau_0)$ in (54) and (55):

$$\begin{aligned}
\frac{1}{\pi} &= 2i\frac{1 - N\cdot(c_0\tau_0 + d_0)^2}{2c_0(c_0\tau_0 + d_0)} g_0(z_0)\bigl(u_0(z)^2\bigr)'\big|_{z=z_0} \\
&\quad - 2i\frac{c_0\tau_0 + d_0}{c_0} g_1(z_0)\bigl(u_0(z)^2\bigr)\big|_{z=z_0}.
\end{aligned} \tag{56}$$

This is a Ramanujan-type series for $1/\pi$.

Remark 13. There are two places, where the use of modularity is crucial: the algebraicity of $z_0 = z(\tau_0)$ (Lemma 7) and the algebraicity of $g_1(z) = \widetilde{v}/u_0^2$ (Lemma 6). The fact that τ is algebraic (and even quadratic) follows from $\gamma_0\tau_0 = N\tau_0$, while the algebraicity of $g_0(z)$ is a purely analytic fact (see the analytic proof of Corollary

in Section 1). It would be nice to avoid the modularity completely, thus providing a purely differential equation proof of equality (56).

It seems that there exist analogous algebraic relations in the case of Picard–Fuchs fourth and fifth order linear differential equations considered above.

The differential equation (28) in Example 3 and its analytic solution $w_0 = w_0(z)$, which is a ${}_5F_4$ hypergeometric series, are related to the following formulas for $1/\pi^2$ proved recently by J. Guillera [8], [9]:

$$\sum_{n=0}^{\infty} \binom{2n}{n}^5 (20n^2 + 8n + 1)\left(-\frac{1}{2^{12}}\right)^n = \frac{8}{\pi^2}, \tag{57a}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^5 (820n^2 + 180n + 1)\left(-\frac{1}{2^{20}}\right)^n = \frac{128}{\pi^2}. \tag{57b}$$

Namely, following the notations in Example 3, for the two specializations of z,

(a) $z = -1/2^{12}$, and
(b) $z = -1/2^{20}$,

we discovered experimentally that

$$3\tau_1 + 4\tau_2 + 4\tau_3 - 2(\tau_1\tau_3 - \tau_2^2) = 14, \tag{58a}$$

$$(\tau_1 - 2)\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} - \tau_2 = \sqrt{5} + 1, \tag{59a}$$

and

$$7\tau_1 + 12\tau_2 + 4\tau_3 - 2(\tau_1\tau_3 - \tau_2^2) = 78, \tag{58b}$$

$$(\tau_1 - 2)\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} - \tau_2 = \sqrt{41} + 3, \tag{59b}$$

respectively, where $\tau_i = \tau_i(z)$ are defined as in Example 3. (One may also verify that, for an arbitrary $z < 0$, we have non-holomorphic relations $\mathrm{Re}(\tau_1/2 - 1) = 0$, $\mathrm{Im}(\tau_1/2 + \tau_2) = 0$, $\mathrm{Re}(\tau_1/4 + \tau_2 + \tau_3 - 1) = 0$, and $\mathrm{Re}(\mathrm{d}\tau_2/\mathrm{d}\tau_1 + 1/2) = 0$.) Equations (58a) and (58b) show that the points $\mathrm{T} = \left(\begin{smallmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{smallmatrix}\right)$ lie on certain rational Humbert surfaces; the equations can be written as

$$\det(\gamma^{(\mathrm{a})}\mathrm{T}) = -5 \quad \text{or} \quad \gamma_1(\gamma^{(\mathrm{a})}\mathrm{T}) = \frac{1}{5}\gamma^{(\mathrm{a})}\mathrm{T}, \tag{60a}$$

$$\det(\gamma^{(\mathrm{b})}\mathrm{T}) = -41 \quad \text{or} \quad \gamma_1(\gamma^{(\mathrm{b})}\mathrm{T}) = \frac{1}{41}\gamma^{(\mathrm{b})}\mathrm{T}, \tag{60b}$$

respectively, where

$$\gamma^{(\mathrm{a})} = \begin{pmatrix} 1 & 0 & -2 & 1\\ \frac{1}{2} & 1 & 0 & -1\\ 0 & 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma^{(\mathrm{b})} = \begin{pmatrix} 1 & 0 & -2 & 3\\ \frac{1}{2} & 1 & 2 & -2\\ 0 & 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are matrices in $\mathrm{Sp}_4(\mathbb{Q})$, and γ_1 is defined in (29). Relations (60a) and (60b) may be viewed as an analogue of (53), although the matrices $\gamma^{(\mathrm{a})}$ and $\gamma^{(\mathrm{b})}$ do not belong to the monodromy group of Example 3.

The equalities (58a) and (58b) happen to hold for the non-holomorphic embedding Z in Section 3 as well: one simply replaces the corresponding entries of T by Z.

We have verified the five other examples in [9] and conclude that the algebraicity seems to appear in all Guillera's identities for $1/\pi^2$ (both conjectural and proved). What is a theoretic background for these algebraic relations?

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Appendix. ON A SUBGROUP OF INFINITE INDEX IN $\mathrm{Sp}_4(\mathbb{Z})$

VICENŢIU PAŞOL

Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ be the two generators for $\mathrm{SL}_2(\mathbb{Z})$.

Let Γ be the subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ generated by the matrices

$$\gamma_0 = \begin{pmatrix} -STS & M \\ 0 & T \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix},$$

where $M = \left(\begin{smallmatrix} 4 & 2 \\ -2 & 1 \end{smallmatrix}\right)$. These are exactly the matrices in (29).

Theorem VP1. *The group* Γ *has infinite index in* $\mathrm{Sp}_4(\mathbb{Z})$.

The idea of the proof is to find a principal $\mathrm{Sp}_4(\mathbb{Z})$-module such that Γ has infinitely many orbits.

For a vector $\mathbf{a} = (a, b, c, d) \in \mathbb{Z}^4$, we say it is reduced if it is primitive (i.e., $\gcd(a, b, c, d) = 1$) and it satisfies the following conditions:

(1) $a \ge 0$, $b \le 0$, $c \ge 0$, $d \ge 0$,
(2) $-b \le d/2 - c$, and
(3) $c \le a/2 + b$.

In particular, we have

(4) $-b \le a/2$ and
(5) $c \le d/2$.

On the set of primitive vectors $(\mathbb{Z}^4)'$ we introduce the involution ε by the rule

$$\varepsilon(a, b, c, d) := (d, -c, -b, a).$$

One easily check that ε preserves the reduced vectors.

Finally, we introduce an algorithm that produces reduced vectors starting with any primitive one.

Algorithm. Fix a pair $\mathbf{a} = (a, b, c, d) \in (\mathbb{Z}^4)'$. Consider the following process.
Step 0. Put $\mathbf{a}_0 = (a_0, b_0, c_0, d_0) := (|a|, -|b|, |c|, |d|)$.
Step 1. Let $b_1 := -\min_{n\in\mathbb{Z}} |b_0 + na_0|$. Put $\mathbf{a}_1 := (a_0, b_1, c_0, d_0)$.
Step 2. Let $c_1 := \min_{n\in\mathbb{Z}} |c_0 + nd_0|$. Put $\mathbf{a}_2 := (a_0, b_1, c_1, d_0)$.
Step 3. Let $n_0 \in \mathbb{Z}$ such that $\min_{n\in\mathbb{Z}} |b_1 + n(d_0 + 2c_1)| = |b_1 + n_0(d_0 + 2c_1)|$. Put $b_2 := -|b_1 + n_0(d_0 + 2c_1)|$, $a_2 = |a_0 + n_0(4c_1 - 2d_0)|$, and $\mathbf{a}_3 := (a_2, b_2, c_1, d_0)$.
Step 4. Let $n_1 \in \mathbb{Z}$ such that $\min_{n\in\mathbb{Z}} |c_1 + n(-a_2 + 2b_2)| = |c_1 + n_1(a_2 + 2b_2)|$. Put $c_2 := |c_1 + n_1(-a_2 + 2b_2)|$, $d_2 = |d_0 - n_1(2a_2 + 4b_2)|$, and $\mathbf{a}_4 := (a_2, b_2, c_2, d_2)$.

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Step 5. If $\mathbf{a}_4 \neq \mathbf{a}$ repeat from Step 1 starting with $\mathbf{a}_4$.

The process must ends by the well order principle. We call the end result by $r(\mathbf{a})$.

Lemma VP1. *We have the following properties for this process:*

(1) $r(\varepsilon(\mathbf{a})) = \varepsilon(r(\mathbf{a}))$,
(2) $r(a,b,c,d) = r(\pm a, \pm b, \pm c, \pm d) = r(r(a,b,c,d))$,
(3) $\mathbf{a}$ *is reduced if and only if* $r(\mathbf{a}) = \mathbf{a}$.

Proof. The first two properties are obvious from the definitions.

For the third one, if $\mathbf{a}$ is reduced, none of Steps 0–4 of the algorithm produce anything new (easy check). Therefore, the process ends and $r(\mathbf{a}) = \mathbf{a}$.

Conversely, if $r(\mathbf{a}) = \mathbf{a}$, we must have that none of Steps 0–4 of the algorithm produce anything new, since the algorithm decreases (not necessarily strict) the absolute value of the components at each step. Conditions (1)–(5) in the definition of a reduced vector are easily checked to be satisfied. □

Lastly, we need a new definition. For two primitive vectors $\mathbf{a}_1, \mathbf{a}_2$ we say they are equivalent, $\mathbf{a}_1 \sim \mathbf{a}_2$, if $r(\mathbf{a}_1) = r(\mathbf{a}_2)$ or $r(\mathbf{a}_1) = \varepsilon(r(\mathbf{a}_2))$. Notice that this relation is an equivalence relation by Lemma VP1.

We give now the main result which implies Theorem VP1.

We let $\mathrm{GL}_4(\mathbb{Z})$ act on $(\mathbb{Z}^4)'$ by multiplication on the left, so the vectors are considered as column vectors.

Let $\widetilde{\Gamma}$ be the subgroup of $\mathrm{GL}_4(\mathbb{Z})$ generated by the following matrices:

$$A = \begin{pmatrix} -STS & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} E & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} E & -ST^{-1}SMT^{-1} \\ 0 & E \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \gamma_1,$$

where E is the 2×2 identity matrix. Since $ACB = \gamma_0$, we do have $\Gamma \subseteq \widetilde{\Gamma}$.

Lemma VP2. *We have* $\mathbf{a} \sim \gamma\mathbf{a}$ *for all* $\gamma \in \widetilde{\Gamma}$.

Proof. It is obvious that it is enough to prove this only for the generators of $\widetilde{\Gamma}$.

If $\gamma = \gamma_1$ we have

$$r(S\mathbf{a}) = r(\varepsilon(\mathbf{a}))) = \varepsilon(r(\mathbf{a})).$$

If $\gamma = A$ we have

$$r\left(A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\right) = r\left(\begin{pmatrix} a \\ b-a \\ c \\ d \end{pmatrix}\right) = r(\mathbf{a}).$$

The last equality follows from the fact that after Step 1 both quantities are equal, i.e.,

$$\min_{n\in\mathbb{Z}} |b + na| = \min_{n\in\mathbb{Z}} |na \pm (b-a)|.$$

If $\gamma = B$, same but for c and d reasoning.

If $\gamma = C$, Step 3 takes care of the equality. $\square$

Lemma VP3. *If* $\mathbf{a}_1, \mathbf{a}_2$ *are reduced and* $\mathbf{a}_1 \neq \mathbf{a}_2$ *and* $\mathbf{a}_1 \neq \varepsilon(\mathbf{a}_2)$*, then* $\mathbf{a}_1 \not\sim_\Gamma \mathbf{a}_2$*.*

Proof. The fact the vectors are reduced implies that they are equal to their reduced vectors. The condition in the corollary implies that $\mathbf{a}_1 \not\sim \mathbf{a}_2$, therefore, by the previous result we cannot have equivalence under $\widetilde{\Gamma}$, in particular under Γ. $\square$

To end the proof of our main result, we just have to observe that we have infinite number of vectors which are reduced and nonequivalent. For example, take the vectors $\mathbf{a}_{p,q} := (p, -1, 0, q)$ for $p, q \geq 2$ any two positive integers.

In this case, since $\mathrm{Sp}_4(\mathbb{Z})$ acts transitively on the set of primitive vectors in $\mathbb{Z}^4$ (so $(\mathbb{Z}^4)'$ is the principal $\mathrm{Sp}_4(\mathbb{Z})$-module we are considering), let $\gamma_{p,q} \in \mathrm{Sp}_4(\mathbb{Z})$ such that $\gamma_{p,q} \cdot {}^t(1,0,0,0) = \mathbf{a}_{p,q}$. Then $\gamma_{p,q} \not\sim_\Gamma \gamma_{p',q'}$ for any $(p,q) \neq (p',q')$. Otherwise, $\mathbf{a}_{p',q'} = \gamma \cdot \mathbf{a}_{p,q}$ for some $\gamma \in \Gamma$. By Lemma VP3 this is impossible.

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